EE 8215: Nonlinear Systems
Lecture 7: Feb 11, 2016, Thu

> Last time: - Poincaré–Bendixon Thm
  - Supercritical Hopf bifurcation
    \[ \dot{x} = ax - r^2 \\
    \dot{y} = 1 \]

> Today: - Subcritical Hopf bifurcations
  - Scaling of equations
  - Center manifold theory

> Subcritical Hopf Bifurcations

\[ \dot{x} = cx + r^3 - r^5 \]
\[ \dot{y} = 1 \]
\[ \Rightarrow \dot{x} = cx + r^3 - r^5 = f(r) \]

\[ \text{Solve: } f(r) = 0 \Rightarrow r^4 - r^2 - \alpha = 0 \]
\[ r = \frac{1 \pm \sqrt{1 + 4\alpha}}{2} \]
\[ \alpha = -\frac{1}{4} \Rightarrow r = \frac{1}{\sqrt{2}} \]

In the phase-plane:
(a) If \( \alpha < -\frac{1}{4} \)
(b) \( -\frac{1}{4} < \alpha < 0 \)
(c) \( \alpha > 0 \)

Bad: always goes to a limit cycle \( \rightarrow \) eg: @ 70mph, no matter where you start, end up @ 70mph - ticket!!
Scaling of Equations (Non-dimensional form of equations)

\[ \frac{dx_1}{dt} = -\alpha x_1 + \beta x_2 \]
\[ \frac{dx_2}{dt} = \frac{y x_1}{\delta + x_1^2} - \eta x_2 \]

Let \( z_1 = \frac{x_1}{X_1} \), \( z_2 = \frac{x_2}{X_2} \), \( \tau = \frac{t}{T} \) — non-dimensional

Objective: write eqns as \( \frac{d}{d\tau} = f(z) \)

\[ \frac{dz_1}{d\tau} = \frac{1}{X_1} \frac{dx_1}{d\tau} = \frac{1}{X_1} \frac{dt}{d\tau} \cdot \frac{dx_1}{dt} = \frac{T}{X_1} \left( -\alpha x_1 + \beta x_2 \right) = \frac{T}{X_1} (-\alpha X_1 z_1 + \beta X_1 z_2) \]

\[ \frac{dz_2}{d\tau} = \frac{1}{X_2} \frac{dx_2}{d\tau} = \frac{1}{X_2} \frac{dt}{d\tau} \cdot \frac{dx_2}{dt} = \frac{T}{X_2} \left( \frac{y x_1}{\delta + x_1^2} - \eta x_2 \right) = \frac{T}{X_2} \left( \frac{y X_1 z_1}{\delta + X_1^2 z_1^2} - \eta X_2 z_2 \right) \]

If \( X_1 \) and \( T \) are properly selected, can bring system to the form:

\[ \frac{dz_1}{d\tau} = -az_1 + z_2 \quad \frac{dz_2}{d\tau} = \frac{z_1}{1+z_1^2} - bz_2 \]

(down to only two parameters: \( a \) & \( b \))

Center Manifold Theory

- Nonlinear system: \( \dot{x} = f(x) \) — order of system

let \( f(0) = 0 \) \( \Rightarrow \) \( \bar{x} = 0 \) is an eqn pt.

let \( A = \frac{\partial f}{\partial x} \bigg|_{x=0} \) have \( k \) eigenvalues on the jw axis

\( n-k \) eigenvalues in the LHP

Linearisation not helpful

Challenge: cannot say anything about \( \bar{x} = 0 \) using linearisation

Write (1) as: \( \dot{x} = Ax + \tilde{f}(x) \) (possible \( \checkmark \))

If we use Taylor series around \( \bar{x} = 0 \)

\( f(x) = f(0) + \frac{\partial f}{\partial x} \bigg|_{x=0} \cdot x + h.o.t \)

\[ \tilde{f}(x) = f(x) - \frac{\partial f}{\partial x} \bigg|_{\bar{x}=0} \cdot x \]

Properties of \( \tilde{f} \):

\[ \tilde{f}(0) = f(0) - \frac{\partial f}{\partial x} \bigg|_{\bar{x}=0} \cdot 0 = 0 \]
\[ \frac{\partial \tilde{f}}{\partial x} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial x} \bigg|_{\bar{x}=0} \Rightarrow \frac{\partial \tilde{f}}{\partial x} \bigg|_{\bar{x}=0} = \frac{\partial f}{\partial x} \bigg|_{\bar{x}=0} - \frac{\partial f}{\partial x} \bigg|_{\bar{x}=0} = 0 \]

Summary: Can write (1) as:

\[ \dot{x} = Ax + \tilde{f}(x) \]
\[ \tilde{f}(0) = 0 \]
\[ \frac{\partial \tilde{f}}{\partial x} \bigg|_{\bar{x}=0} = 0 \]
Introduce change of coordinates:

\[ \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{such that} \quad \begin{cases} y = Ax + g_1(y, z) \\ z = Az + y, g_2(y, z) \end{cases} \]

where \( A_1 \) contains eigenvalues on the \( j\0 \) axis
\( A_2 \) contains eigenvalues in the LHP

\[ \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} g_1(y, z) \\ g_2(y, z) \end{bmatrix} \]

**Thm:** There is an invariant manifold \( z = h(y) \) in the neighbourhood of the origin that satisfies \( h(0) = 0 \) and \( \frac{\partial h}{\partial y} \bigg|_{y=0} = 0 \)

Geometric interpretation:

- invariant \( \Rightarrow \) start there, stay there
- manifold: surface in a higher dimensional state

Study 1st eqn (reduced-order system) \( w = z = h(y) \), then stability properties of the reduced-order system determine stability properties of the whole system.

**Main result**

**Thm:** If the origin of the reduced system \( \dot{y} = Ay + g_1(y, h(y)) \) is asymptotically stable (or unstable) then: the origin of \( \bigcirc \) is asymptotically stable (or unstable)

**Key challenge:** characterise \( h(y) \) \( \Leftrightarrow \) center manifold

Introduce \( w = z - h(y) \)

If \( w = 0 \Rightarrow \dot{w} = 0 \) (invariant manifold)

\[ \dot{w} = \dot{z} - \dot{h} = \dot{z} - \frac{\partial h}{\partial y} \dot{y} = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y} [Ay + g_1(y, h(y))] = 0 \]

Eqn that characterises a manifold:

\[ A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y} [Ay + g_1(y, h(y))] = 0 \]

**Eq (3):** Let \( y(\theta) \in \mathbb{R} \) and look for approximate solutions of (2)

Use Taylor series of \( h(y) \) around \( 0:0 \)

\[ h(y) = h(0) + \frac{\partial h}{\partial y} \bigg|_0 y + \frac{\partial^2 h}{\partial y^2} \bigg|_0 y^2 + \frac{\partial^3 h}{\partial y^3} \bigg|_0 y^3 + \mathcal{O}(y^4) \]

from earlier:

\[ \begin{bmatrix} \frac{\partial h}{\partial y} \\ \frac{\partial^2 h}{\partial y^2} \\ \frac{\partial^3 h}{\partial y^3} \end{bmatrix} \bigg|_0 \]

locally, only quadratic \& h.o.t.

\( \therefore \) already captured linear terms
\[ h(y) = h_2 y^2 + h_3 y^3 + O(h^4) \]

Eq (4):
\[ \begin{align*}
    y &= 0 \cdot y + y^2 z \\
    z &= -z + a y^2 \\
    A_1 &= 0, \\
    A_2 &= -1,
\end{align*} \]

Plug \( g = h(y) = h_2 y^2 + h_3 y^3 + O(h^4) \) in and equate equal orders in \( y \),
turns out \( h_2 = a \rightarrow a > 0 \)
\[ a < 0 \]