A Parametric Uncertainty Analysis Method for Markov Reliability and Reward Models

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Abstract

A common concern with Markov reliability and reward models is that model parameters, i.e., component failure and repair rates, are seldom perfectly known. This paper proposes a numerical method based on the Taylor series expansion of the underlying Markov chain stationary distribution (associated to the reliability and reward models) to propagate parametric uncertainty to reliability and performability indices of interest. The Taylor series coefficients are expressed in closed form as functions of the Markov chain generator-matrix group inverse. Then, in order to compute the probability density functions of the reliability and performability indices, random variable transformations are applied to the polynomial approximations that result from the Taylor series expansion. Additionally, closed-form expressions that approximate the expectation and variance of the indices are also derived. A significant advantage of the proposed framework is that only the parametrized Markov chain generator matrix is required as an input, i.e., closed-form expressions for the reliability and performability indices as a function of the model parameters are not needed. Several case studies illustrate the accuracy of the proposed method in approximating distributions of reliability and performability indices. Additionally, analysis of a large model demonstrates lower execution times compared to Monte Carlo simulations.

Index Terms

Markov reliability models, Markov reward models, parametric uncertainty.

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This work was supported in part by the National Science Foundation (NSF) under grant ECCS-CAR-0954420.

NOMENCLATURE

Λ Markov chain generator matrix

$\theta = [\theta_1, \ldots, \theta_m]$ Vector of generator-matrix parameters

$\pi(\theta) = [\pi_0(\theta), \ldots, \pi_n(\theta)]$ Stationary distribution of the Markov chain

ξ, γ Reward and accumulated reward of a Markov reward model

$\Theta = [\Theta_1, \ldots, \Theta_m]$ Vector of random variables that describes uncertain generator-matrix parameters

$\Delta\Theta$ Zero-mean random vector with the same distribution as Θ

$\Pi = [\Pi_0, \ldots, \Pi_n]$ Vector of random variables that describes the stationary distribution of the Markov chain

Ξ, Γ Random variables that describe reward and accumulated reward of a Markov reward model

$N \sim \mathcal{N}(m_N, \sigma_N^2)$ Normally distributed random variable with mean m_N and variance σ_N^2

$U \sim \mathcal{U}(a_U, b_U)$ Uniformly distributed random variable over the interval $[a_U, b_U]$

e Row vector with all entries equal to 1

I. INTRODUCTION

Parametric uncertainty in Markov reliability and reward models is a significant concern as component failure and repair rates are seldom perfectly known. In lieu of precise numbers, failure and repair rates can be modeled as random variables with distributions determined by various methods, e.g., utilizing known confidence intervals and distributions of aleatory uncertainties such as the mean time to failure [1], applying the maximum entropy principle if only the range of the uncertain parameters is known [2], or based on engineering experience and field data [3], [4]. Given the probability density functions (pdfs) of the uncertain parameters, this paper proposes a framework to compute the pdfs of the Markov reliability model stationary distribution, and Markov reward model performability indices, both for repairable systems in which the underlying Markov chains are ergodic[1].

[1]A Markov chain is said to be ergodic if it has a unique stationary distribution independent of initial conditions [5].

Set-theoretic methods are an alternative to address the problem of parametric uncertainty in Markov models. For instance, methods based on interval-arithmetic (see, e.g., [6]) have been proposed in [7], [8]. In such works, it is assumed that no model parameter statistics are available, but instead, model parameters are constrained to lie in a set (perhaps centered about a nominal value), which is then propagated by set operations through the Markov model to relevant indices. This approach represents a worst-case uncertainty analysis, as no parameter distributions are assumed due to a lack of statistical information. In this regard, such methods are conceptually very different from the method proposed in this paper, as we adopt a probabilistic model for the values that the parameters can take.

The proposed framework involves the use of Taylor series expansions to approximate the entries of the Markov chain stationary distribution vector, i.e., the steady-state occupational probabilities of different states, as polynomial functions of the uncertain parameters, which are modeled as random variables. A significant contribution of this work is the derivation of the Taylor series coefficients, which are expressed in closed form as functions of the generator-matrix group inverse [9]. Subsequently, random variable transformations are applied to numerically compute the pdfs of the Markov chain steady-state probabilities and performability indices. Additionally, closed-form expressions for the expectation and variance of these indices are derived from a direct analysis of lower-order approximations of the Taylor series expansion. Note that if closed-form expressions for the relevant indices as a function of the model parameters were readily available, Taylor series expansions would be unnecessary; however, in general, it is difficult to obtain these expressions in closed form.

The use of Taylor series expansions to study parametric uncertainty in Markov reward models has been proposed in [2], [10]. In these works, the Taylor series coefficients are expressed in terms of the inverse of the underlying Markov chain generator matrix. However, since the generator matrix of ergodic Markov chains is singular, it is unclear how the ideas in [2], [10] can be implemented in practice. Additionally, while the approach is sketched out, it is not applied directly in the case studies. Methods to propagate uncertainty based on the Markov chain transient solution sensitivity to model parameters are outlined in [4], [11], [12], and [13].

By contrast, since we focus on repairable systems, our method focuses directly on the stationary distribution of ergodic Markov chains which are used to model repairable systems. Note also that our framework not only proposes closed-form approximations for the expectation and variance of reliability and performability indices, but also provides a numerical method to derive the pdfs of these indices. The sensitivities could be computed following alternative methods (see e.g., [4], [11] and the references therein) before applying the techniques proposed here to obtain the pdfs of the reliability/performability indices. Finally, a significant advantage of the proposed framework is that the only required input is the Markov chain generator matrix, i.e., closed-form expressions for the stationary distribution and performability indices as a function of the model parameters are not required a priori.

We demonstrate the application of the proposed framework in analyzing Markov reliability and reward models with several case studies, including: i) a two-state model for a single component with two operating modes, ii) a three-state model for a two-component load-sharing system with common-cause failures, and iii) an $n+1$ state model for n components, each with two operating modes. In the first two case studies, we illustrate the accuracy of the proposed method by comparing results with Monte Carlo simulations (and the exact analytical result when available). The expectation and variance derived from the analytical expressions are also compared with those obtained numerically from the derived pdfs. The final case study compares the execution time of the proposed approach with Monte Carlo simulations to compute the pdf of a particular performability metric. The execution time of the proposed Taylor series method is noted to be lower than Monte Carlo simulations for large models as long as there are a few uncertain parameters.

The remainder of this paper is organized as follows. In Section II, we introduce Markov reliability and reward model fundamentals, and present closed-form expressions for the sensitivity of the Markov chain stationary distribution to model parameters as a function of the generator-matrix group inverse. In Section III, we outline the numerical framework to compute the pdfs of the stationary distribution and performability indices, and also provide pseudocode for computer implementation of the framework. Case studies are presented in Section IV, and concluding remarks are provided in Section V.

II. Preliminaries

A brief introduction to the fundamentals of Markov reliability and reward models, and the group inverse of ergodic Markov chains is presented in this section. Interested readers are referred to [14], [15], and [9], respectively, for a more detailed account on these topics.

A. Markov Reliability Models

Let $X = \{X(t),\, t \geq 0\}$ denote a stochastic process taking values in a countable set $\mathcal{S}$. The stochastic process X is called a continuous-time Markov chain if it satisfies the so called Markov property, which is to say that

$$\Pr\{X(t_n) = i | X(t_{n-1}) = j_{n-1}, \ldots, X(t_1) = j_1\} = \Pr\{X(t_n) = i | X(t_{n-1}) = j_{n-1}\}\,, \qquad (1)$$

for all $i,\, j_1, \ldots,\, j_{n-1} \in \mathcal{S}$, and $t_1 < \ldots < t_n$ [16]. The chain X is said to be *homogeneous* if it satisfies

$$\Pr\{X(t) = i | X(s) = j\} = \Pr\{X(t-s) = i | X(0) = j\}\,,\ \forall i,\, j \in \mathcal{S},\, 0 < s < t. \qquad (2)$$

Homogeneity of X implies that the times between transitions are exponentially distributed. The chain X is said to be *irreducible* if for every pair i, j of states, we have $\Pr\{X(t) = i | X(0) = j\} > 0$ for some $t > 0$, i.e., every state in an irreducible chain is accessible from every other state.

In this paper, we consider the class of continuous-time Markov chains that are homogeneous, irreducible, and take values in a finite set $\mathcal{S} = \{0,\, 1,\, 2, \ldots,\, n\}$, where $0,\, 1,\, 2, \ldots,\, n-1$ index system configurations that arise due to component faults, and n indexes the nominal, non-faulty configuration. Let X denote a chain belonging to this class; then since X is irreducible and takes values in a finite set, it follows that X is also *ergodic*, i.e., it has a unique stationary distribution independent of initial conditions [15]. Let $\tilde{\pi}_i(t),\, t \geq 0$, be the probability that the chain is in state

i at time t, and define the corresponding probability vector as $\tilde{\pi}(t) = [\tilde{\pi}_0(t),\, \tilde{\pi}_1(t), \ldots,\, \tilde{\pi}_n(t)]$. The evolution of $\tilde{\pi}(t)$ is governed by the Chapman-Kolmogorov equations:

$$\dot{\tilde{\pi}}(t) = \tilde{\pi}(t)\Lambda, \tag{3}$$

with $\tilde{\pi}_n(0) = 1$, $\tilde{\pi}_i(0) = 0$, $i = 0, 1, \ldots, n-1$, and where $\Lambda \in \mathbb{R}^{n+1\times n+1}$ is the Markov chain generator matrix whose entries are a function of component failure and repair rates [14]. By construction, all the row sums in Λ are zero, which implies that Λ is not invertible. The steady-state solution of (3) is referred to as the stationary distribution of the chain; it is denoted by π, and is obtained as the solution of

$$\pi\Lambda = 0, \quad \pi e^T = 1, \tag{4}$$

where $e \in \mathbb{R}^{n+1}$ is a row vector with all entries equal to one. The stationary distribution of an ergodic Markov chain is unique (independent of initial conditions), and a function of the generator-matrix parameters (interchangeably referred to as model parameters) which are denoted by θ_j, $j = 1, 2, \ldots, m$. To explicitly represent parametric dependence, the generator matrix and stationary distribution are expressed as $\Lambda(\theta)$ and $\pi(\theta) = [\pi_0(\theta), \ldots, \pi_n(\theta)]$, respectively, where $\theta = [\theta_1, \theta_2, \ldots, \theta_m]$.

B. Markov Reward Models

The terminology in this section is adopted from [15]. A Markov reward model is defined by a Markov chain taking values in a finite set $\mathcal{S}$ and a *reward function* $\varrho :\ \mathcal{S} \to \mathbb{R}$ that maps each element $i \in \mathcal{S}$ into a real-valued quantity ρ_i, which captures some performance metric of interest while in state i. At time t, the value that ϱ takes can be described by a random variable $P(t)$. The *instantaneous reward*, denoted by $\xi(t)$, is a probabilistic measure of system performance given by the expected value of $P(t)$:

$$\xi(t) \equiv \mathrm{E}[P(t)] = \sum_{i=0}^{n} \tilde{\pi}_i(t)\rho_i = \tilde{\pi}(t)\rho^T, \tag{5}$$

where $\rho = [\rho_0, \rho_1, \ldots, \rho_n]$. The *reward*, denoted by ξ, is a long-term measure of system performance, and it is defined as

$$\xi \equiv \lim_{t\to\infty} \xi(t) = \lim_{t\to\infty} \sum_{i=0}^{n} \tilde{\pi}_i(t)\rho_i = \sum_{i=0}^{n} \pi_i \rho_i = \pi\rho^T, \tag{6}$$

where $\pi = [\pi_0, \pi_1, \ldots, \pi_n]$ is the Markov chain stationary distribution. If the values that the reward function ϱ takes are defined in per-unit time, then ξ describes the average rate at which the system delivers/consumes some quantity that captures a measure of system performance. The *accumulated reward*, denoted by γ, is a quantity measuring system performance in a period $[0, \tau]$, and it is defined as

$$\gamma \equiv \int_0^\tau \mathrm{E}[P(t)]dt = \int_0^\tau \tilde{\pi}(t)\rho^T dt. \tag{7}$$

Let t_0 be the time at which the effect of initial conditions in (3) has vanished, i.e., $\tilde{\pi}(t) \approx \pi \,\forall t \geq t_0$. Then, for $\tau \gg t_0$, it follows that [17]:

$$\gamma = \int_0^\tau \tilde{\pi}(t)\rho^T dt = \int_0^{t_0} \tilde{\pi}(t)\rho^T dt + \int_{t_0}^{\tau} \tilde{\pi}(t)\rho^T dt \approx \int_0^{t_0} \tilde{\pi}(t)\rho^T dt + \pi\rho^T(\tau - t_0). \tag{8}$$

Now, by applying the mean-value theorem for integration to the first term of the last equality above, we obtain

$$\gamma \;\approx\; \tilde{\pi}(s)\rho^T t_0 + \pi\rho^T(\tau - t_0) = \left(\tilde{\pi}(s)\rho^T - \pi\rho^T\right) t_0 + \pi\rho^T\tau \approx \pi\rho^T\tau, \tag{9}$$

where $\tilde{\pi}(s) = \tilde{\pi}(t)|_{t=s}$ for some $s \in [0, t_0]$. Since $0 \leq \tilde{\pi}_i(s) \leq 1$ and $0 \leq \pi_i \leq 1$, $\forall i = 0, 1, \ldots, n$, and $\tau \gg t_0$, the term $\pi\rho^T\tau$ dominates $\left(\tilde{\pi}(s)\rho^T - \pi\rho^T\right) t_0$, and as a result, $\gamma \approx \pi\rho^T\tau$. To explicitly represent the dependence of the reward and accumulated reward on the generator-matrix parameters, they are expressed as $\xi(\theta)$ and $\gamma(\theta)$, respectively.

C. Numerical Computation of the Stationary Distribution and the Group Inverse

For ergodic Markov chains, the generator-matrix group inverse enables the numerical calculation of $\partial^k \pi_i(\theta)/\partial\theta_j^k$, $i = 0, 1, \ldots, n$; $j = 1, 2, \ldots, m$; $k > 0$, as will be discussed in Section

II-D. The group inverse of $\Lambda = \Lambda(\theta)$ for some θ is denoted by $\Lambda^{\#}$, and is given by the unique solution of

$$\begin{cases} \Lambda\Lambda^{\#}\Lambda = \Lambda, \\ \Lambda^{\#}\Lambda\Lambda^{\#} = \Lambda^{\#}, \\ \Lambda\Lambda^{\#} = \Lambda^{\#}\Lambda, \end{cases} \tag{10}$$

if and only if $rank(\Lambda) = rank(\Lambda^2)$, which is a condition that always holds for generator matrices of ergodic Markov chains [18]. A number of techniques amenable for computer implementation have been proposed to compute the group inverse [9]. An approach involving the QR factorization of Λ yields the stationary distribution $\pi = \pi(\theta)$, and the group inverse, $\Lambda^{\#}$ [19]. In this method, Λ is factored as $\Lambda = QR$, where, $Q,\, R \in \mathbb{R}^{n+1 \times n+1}$. The matrix R is of the form

$$R = \begin{bmatrix} U & -Ue^T \\ 0 & 0 \end{bmatrix}, \tag{11}$$

where $U \in \mathbb{R}^{n \times n}$ is a nonsingular upper-triangular matrix, and $e \in \mathbb{R}^n$ is a row vector with all entries equal to one. The stationary distribution can be derived by normalizing the last column of Q:

$$\pi_j = \frac{q_{j+1,n+1}}{\sum\limits_{i=1}^{n+1} q_{i,\,n+1}},\; j = 0,\, 1, \ldots,\, n. \tag{12}$$

The group inverse is related to Q and R as follows:

$$\Lambda^{\#} = (I - e^T\pi) \begin{bmatrix} U^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^T (I - e^T\pi). \tag{13}$$

D. Sensitivity of the Stationary Distribution and Reward to Model Parameters

For Markov reliability models, the first-order sensitivity of stationary distributions to model parameters was derived in [17]. We extend the ideas in [17] to obtain higher-order sensitivities, which is a key result utilized extensively in the framework proposed subsequently. The k-order sensitivity of the stationary distribution $\pi(\theta_1, \theta_2, \ldots, \theta_m)$ to the i parameter θ_i is given by

$$\frac{\partial^k \pi(\theta)}{\partial \theta_i^k} = k!\,(-1)^k\,\pi(\theta)\left(\frac{\partial \Lambda}{\partial \theta_i}\Lambda^{\#}\right)^k. \tag{14}$$

The derivation of (14) is included in the Appendix. From (6) and (14), the sensitivity of the reward to the i parameter can be expressed as

$$\frac{\partial^k \xi(\theta)}{\partial \theta_i^k} = \frac{\partial^k \pi(\theta)}{\partial \theta_i^k}\rho^T = k!\,(-1)^k\,\pi(\theta)\left(\frac{\partial \Lambda}{\partial \theta_i}\Lambda^{\#}\right)^k \rho^T. \tag{15}$$

III. Computation of PDFs for the Markov Chain Stationary Distribution and Reward Models

In this section, we propose numerical methods to compute the pdfs of the stationary distribution, the reward, and the accumulated reward, given the parametrized Markov chain generator matrix and the model-parameter pdfs. First, we demonstrate how the pdfs of the stationary distribution can be derived for the case where a single parameter in the generator matrix is uncertain. Then, for the more general multiple-parameter case, we leverage the results of the single-parameter case to show how the pdfs of the stationary distribution, the reward, and the accumulated reward can be computed.

Let $\Theta = [\Theta_1, \Theta_2, \ldots \Theta_m]$ be the vector of random variables that describes the model parameters, and let $f_{\Theta_j}(\theta_j)$ denote the pdf of Θ_j, $j = 1, 2, \ldots, m$. It is assumed that the Θ_j's are independent continuous random variables with known pdfs. Therefore, the steady-state probabilities are random variables that can be collectively described by a random vector $\Pi = [\Pi_0, \Pi_1, \ldots, \Pi_n]$, where $\Pi_i = \pi_i(\Theta)$. Similarly, the reward, $\Xi = \xi(\Theta)$, and the accumulated reward, $\Gamma = \gamma(\Theta) = \Xi \cdot \tau$, are random variables with pdfs $f_\Xi(\xi)$ and $f_\Gamma(\gamma)$, respectively.

If closed-form expressions for the stationary distribution as a function of the model parameters were available and if the expressions were invertible, $f_{\Pi_i}(\pi_i)$, $f_\Xi(\xi)$, and $f_\Gamma(\gamma)$ could be determined through the well-known random-variable-transformation method stated in the following Lemma (see [20] for a complete proof).

Lemma 1. *Consider a random variable X with pdf $f_X(x)$ and a differentiable, real-valued* *function $g(x)$. The pdf of the random variable $Y = g(X)$, $f_Y(y)$, is given by*

$$f_Y(y) = \sum_{i=1}^{r} \frac{f_X(x_i)}{|g'(x_i)|},\ g'(x_i) \equiv \left.\frac{dg}{dx}\right|_{x=x_i} \neq 0, \tag{16}$$

where $x_1,\ x_2, \ldots x_r$ are r real roots of $y = g(x)$.

The main impediment in directly applying the above Lemma to our problem is that it is seldom possible to obtain closed-form expressions for the Markov chain stationary distribution, $\pi(\theta)$ ($g(x)$ in the context of Lemma 1). Furthermore, the number of roots of $y = g(x)$ depends on the value of y, and might not be finite unless $g(x)$ is a polynomial.

In our method, to derive $f_{\Pi_i}(\pi_i)$ and $f_{\Xi}(\xi)$, the functions $\pi_i(\Theta)$ and $\xi(\Theta)$ are first approximated by polynomials by truncating their Taylor series expansions. Since we model these functions as polynomials, we are guaranteed to have a finite number of roots. The Taylor series coefficients are the sensitivities $\partial^k \pi_i(\theta)/\partial\theta^k$ and $\partial^k \xi(\theta)/\partial\theta^k$. In general, obtaining these sensitivities is a difficult task, however, they can be computed from the generator-matrix group inverse as shown in (14)-(15). Once the polynomial characterization is available, Lemma 1 (and its extension to the multiple-parameter case) can be applied to compute $f_{\Pi_i}(\pi_i)$ and $f_{\Xi}(\xi)$ by evaluating the roots of the polynomial approximations, which are easy to obtain numerically. Since the accumulated reward Γ is the product of the reward Ξ, and a constant τ, $f_{\Gamma}(\gamma)$ can be easily expressed as a function of $f_{\Xi}(\xi)$ and τ.

A. Single Parameter Case

Consider the case where a single parameter in the generator matrix is uncertain. This parameter is denoted by θ and described by a random variable Θ, whose pdf $f_{\Theta}(\theta)$, is known[2]. To derive the pdf of the steady-state probability $\Pi_i = \pi_i(\Theta)$, we begin by expressing Θ as

$$\Theta = m_{\Theta} + \Delta\Theta, \tag{17}$$

[2]While we have defined $\theta = [\theta_1,\, \theta_2, \ldots, \theta_m]$ as the vector of model parameters, in this subsection, we abuse notation and denote the single uncertain model parameter by θ.

where m_Θ is the mean of Θ, and $\Delta\Theta$ is a zero-mean random variable such that $f_{\Delta\Theta}(\Delta\theta) = f_\Theta(m_\Theta + \Delta\theta)$. We can expand $\pi_i(\Theta)$ around the mean of Θ using a Taylor series expansion as follows:

$$\Pi_i = \pi_i(m_\Theta + \Delta\Theta) = \pi_i(m_\Theta) + \sum_{k=1}^{\infty} \frac{a_{ki}}{k!} \Delta\Theta^k. \tag{18}$$

The k-order Taylor series coefficient, a_{ki}, follows from (14):

$$a_{ki} \;=\; \left.\frac{d^k \pi_i(\theta)}{d\theta^k}\right|_{\theta=m_\Theta} = k!\,(-1)^k\,\pi(\theta)\left(\frac{d\Lambda(\theta)}{d\theta}\Lambda^{\#}\right)^k e_i^T\Bigg|_{\theta=m_\Theta}, \tag{19}$$

where $e_i \in \mathbb{R}^{n+1}$ is a row vector with a 1 as the i entry and zero otherwise.

1) Probability density function of Π_i*:* Since the exact, analytical, closed-form expression for $\pi_i(\Delta\Theta)$ is not known, to apply the result in Lemma 1, Π_i is first expressed as $\Pi_i = p_i(\Delta\Theta)$, where p_i is a polynomial with real coefficients obtained by truncating the Taylor series in (18) at the t term:

$$\Pi_i = p_i(\Delta\Theta) = \pi_i(m_\Theta) + \sum_{k=1}^{t} \frac{a_{ki}}{k!} \Delta\Theta^k. \tag{20}$$

Then, applying (16), $f_{\Pi_i}(\pi_i)$ can be computed as

$$f_{\Pi_i}(\pi_i) = \sum_{j=1}^{r} \frac{f_{\Delta\Theta}(\Delta\theta_j)}{|p_i'(\Delta\theta_j)|}, \tag{21}$$

where $\Delta\theta_1, \Delta\theta_2, \ldots, \Delta\theta_r$ are the $r \leq t$ real roots of the polynomial equation $\pi_i = p_i(\Delta\theta)$, and

$$p_i'(\Delta\theta_j) \equiv \left.\frac{dp_i(\Delta\theta)}{d\Delta\theta}\right|_{\Delta\theta=\Delta\theta_j} = \sum_{k=1}^{t} \frac{a_{ki}}{(k-1)!} \Delta\theta_j^{k-1}. \tag{22}$$

2) Computer Implementation: Algorithm 1 provides the pseudocode for computer implementation of the method outlined in (17)-(22) to compute $f_{\Pi_i}(\pi_i)$, $i = 0, 1, \ldots, n$. Since (21) has to be evaluated point wise, π_i is appropriately discretized between 0 and 1 in steps of $d\pi_i$ to obtain the vector $\bar{\pi}_i = [0 : d\pi_i : 1]$. A first-order Taylor series expansion can be utilized if the function $\pi_i(\theta)$ is not far from linear within one standard deviation away from the mean, m_Θ [21]. Alternately, higher-order expansions can be utilized. Given the parametrized generator matrix, it is easy to compute $\frac{\partial\Lambda}{\partial\theta}$ and obtain the QR factorization of the generator matrix at the mean of

Θ, $\Lambda(m_\Theta)$. Next $\pi_i(m_\Theta)$ is obtained as shown in (12) by normalizing the last column of Q, the group inverse $\Lambda^\#$ is obtained from (13), and the Taylor series coefficients, a_{ki}, $k = 1, 2, \ldots, t$, are computed using (19). In the for loop, $f_{\Pi_i}(\pi_i)$ is evaluated point wise for each entry of $\bar{\pi}_i$. This involves computing the r real roots of the equation $\bar{\pi}_i(l) = p_i(\Delta\theta)$, $\forall l$, where $\bar{\pi}_i(l)$ is the l entry in $\bar{\pi}_i$,[3] and then applying (21)-(22).

Algorithm 1 Computation of $f_{\Pi_i}(\pi_i)$ for the single-parameter case.

define $\bar{\pi}_i = [0 : d\pi_i : 1]$
define Taylor series order t
compute $\frac{\partial \Lambda}{\partial \theta}$ and $QR = \Lambda(m_\Theta)$
compute $\pi_i(m_\Theta)$ from (12), $\Lambda^\#$ from (13), and a_{ki}, $k = 1, 2, \ldots, t$ from (19)
for $\hat{\pi}_i = 0 : d\pi_i : 1$ **do**
    compute real roots of $\pi_i(m_\Theta) - \hat{\pi}_i + \sum_{k=1}^{t} \frac{a_{ki}}{k!} \Delta\theta^k = 0$, denote them by $\Delta\theta_j$, $j = 1, \ldots, r$
    for $j = 1$ to r **do**
        compute $f_{\Delta\Theta}(\Delta\theta_j)$ and $p_i'(\Delta\theta_j) = \sum_{k=1}^{t} \frac{a_{ki}}{(k-1)!} \Delta\theta_j^{k-1}$
    end for
    compute $f_{\Pi_i}(\hat{\pi}_i) = \sum_{j=1}^{r} \frac{f_{\Delta\Theta}(\Delta\theta_j)}{\left|p_i'(\Delta\theta_j)\right|}$
end for

3) Expectation and Variance of Π_i: While the method outlined in (17)-(22) provides the pdf of the Markov chain stationary distribution, it might be sufficient—for the purpose of back-of-the-envelope calculations—to compute the expected value and variance of Π_i. These could then be used together with Markov and Chebyshev inequalities to get accurate upper bounds on the probabilities of various events of interest [22]. The expected value of Π_i, denoted by m_{Π_i}, can be derived from (20) as

$$m_{\Pi_i} \equiv \mathrm{E}[\Pi_i] = \pi_i(\mu_\Theta) + \sum_{k=1}^{t} \frac{a_{ki}}{k!} \mathrm{E}\left[\Delta\Theta^k\right] . \tag{23}$$

[3] In the pseudocode provided in Algorithms 1, 2, and 3, we denote the entries of the vector $\bar{x}$ by the variable $\hat{x}$

Since the pdf of $\Delta\Theta$ is known, it is easy to compute the expectations, $\mathrm{E}[\Delta\Theta^k]$, $k > 0$. The variance of Π_i, denoted by $\sigma^2_{\Pi_i}$, can be derived from (20) and (23) as

$$\sigma^2_{\Pi_i} \equiv \mathrm{Var}\left(\Pi_i\right) = \sum_{k=1}^{t} \left(\frac{a_{ki}}{k!}\right)^2 \mathrm{Var}(\Delta\Theta^k) + \sum_{k=1}^{t} \sum_{m=1,\, m\neq k}^{t} \frac{a_{ki}}{k!}\frac{a_{mi}}{m!} \cdot \mathrm{Cov}\left(\Delta\Theta^k, \Delta\Theta^m\right), \qquad (24)$$

where $\mathrm{Var}\left(\Delta\Theta^k\right)$, and $\mathrm{Cov}\left(\Delta\Theta^k, \Delta\Theta^m\right)$ are given by

$$\mathrm{Var}(\Delta\Theta^k) = \mathrm{E}\left[\Delta\Theta^{2k}\right] - \left(\mathrm{E}\left[\Delta\Theta^k\right]\right)^2, \qquad (25)$$

$$\mathrm{Cov}(\Delta\Theta^k, \Delta\Theta^m) = \mathrm{E}\left[\Delta\Theta^{k+m}\right] - \mathrm{E}\left[\Delta\Theta^k\right]\mathrm{E}\left[\Delta\Theta^m\right]. \qquad (26)$$

We now present an example that illustrates the ideas described so far.

Example 1. Consider a component with two possible operating states. In state 1, the component performs its intended function, and in state 0, it has failed. The failure rate of the component is denoted by λ, and the repair rate is denoted by μ. The state of the component (functioning or failed) can be described by a two-state Markov chain. The generator matrix for this chain is given by

$$\Lambda = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}. \qquad (27)$$

Denote the stationary distribution of the chain by $\pi = [\pi_0,\, \pi_1]$. By solving (4) with Λ given in (27), we obtain

$$\pi_0 = \frac{\lambda}{\mu+\lambda}, \quad \pi_1 = \frac{\mu}{\mu+\lambda}. \qquad (28)$$

Suppose the failure rate λ is uncertain and described by a normal random variable L with mean, $m_L = 5.5$ and standard deviation, $\sigma_L = 0.5$. The repair rate is assumed to be perfectly known and given by $\mu = 5.5$. Note that failure and repair rates have units of per-unit time. To streamline the presentation, we omit the units in the following discussion. In this example, we focus on how Algorithm 1 can be applied to compute $f_{\Pi_1}(\pi_1)$. Subsequently, we will compare the result obtained from Algorithm 1 with the exact analytical result.

Figure 1 depicts the function $\pi_1 = \frac{\mu}{\mu+\lambda} = \frac{\mu}{\mu+(\Delta\lambda+m_L)}$ and three polynomial approximations ($t = 1, 2, 3$), from which it is clear that a third-order expansion is accurate enough. The QR factorization of $\Lambda(\lambda)$ for $\lambda = m_L = 5.5$ and $\mu = 5.5$, results in:

$$Q = \begin{bmatrix} -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{bmatrix};\ R = \begin{bmatrix} 7.7782 & -7.7782 \\ 0 & 0 \end{bmatrix}. \tag{29}$$

As described in (12), by normalizing the last column of Q, we obtain the stationary distribution $\pi(m_L) = [\pi_0(m_L),\ \pi_1(m_L)] = [0.5,\ 0.5]$. From (11) and (29), $U = 7.7782$. Then, applying (13), we obtain the group inverse:

$$\Lambda^{\#} = \begin{bmatrix} -0.0455 & 0.0455 \\ 0.0455 & -0.0455 \end{bmatrix}. \tag{30}$$

Using (19) to compute the Taylor-series coefficients provides the following third-order polynomial approximation

$$\begin{aligned} p_1(\Delta\lambda) &= \pi_1(m_L) + a_{11}(\Delta\lambda) + \frac{1}{2}a_{21}\,(\Delta\lambda)^2 + \frac{1}{6}a_{31}\,(\Delta\lambda)^3 \\ &= 0.5 - 0.0455(\Delta\lambda) + 0.0041(\Delta\lambda)^2 - 3.7566\text{e-}4(\Delta\lambda)^3. \end{aligned} \tag{31}$$

In order to numerically compute $f_{\Pi_1}(\pi_1)$, $0 \leq \pi_1 \leq 1$, we discretize π_1 as $\bar{\pi}_1 = [0 : 0.0001 : 1]$. We then compute the roots of the equation $\bar{\pi}_1(l) = p_1(\Delta\lambda)$, $\forall l$, where $\bar{\pi}_1(l)$ denotes the l entry of $\bar{\pi}_1$. The real roots are subsequently used in (21) to obtain $f_{\Pi_1}(\pi_1)$. For example, for $\hat{\pi}_1 = 0.5$, the real root is $\Delta\lambda_1 = 0$, and there are two complex roots, $\Delta\lambda_{2,3} = 5.5 \pm 9.5263\,j$ (which are discarded). Since L (and hence ΔL) is normally distributed, it follows that

$$f_{\Delta L}(\Delta\lambda_1) = \frac{1}{\sqrt{2\pi\sigma_L^2}} \exp\left(-\frac{\Delta\lambda_1^2}{2\sigma_L^2}\right) = 0.7979. \tag{32}$$

From (31) we obtain

$$p_1'(\Delta\lambda_1) = \left.\frac{dp_1(\Delta\lambda)}{d\Delta\lambda}\right|_{\Delta\lambda=\Delta\lambda_1} = -0.0455 + 0.0082(\Delta\lambda_1) - 0.0011(\Delta\lambda_1)^2 = -0.0455. \tag{33}$$

Substituting (32) and (33) in (21), we get $f_{\Pi_1}(\hat{\pi}_1 = 0.5) = 17.5363$. This procedure is repeated for all other entries of $\bar{\pi}_1$ and the results are plotted in Fig. 2.

We can also compare the numerical solution with the exact solution obtained by applying random-variable transformation to the function $\Pi_1 = \mu / (\mu + L)$, which results in

$$f_{\Pi_1}(\pi_1) = f_L(\tilde{\lambda}) \frac{\left(\tilde{\lambda} + \mu\right)^2}{\mu}; \; \tilde{\lambda} = \frac{\mu(1 - \pi_1)}{\pi_1}. \tag{34}$$

Figure 2 also depicts $f_{\Pi_1}(\pi_1)$ computed using the exact analytical expression in (34). The results show a very good match between the approximation and the exact solution. □

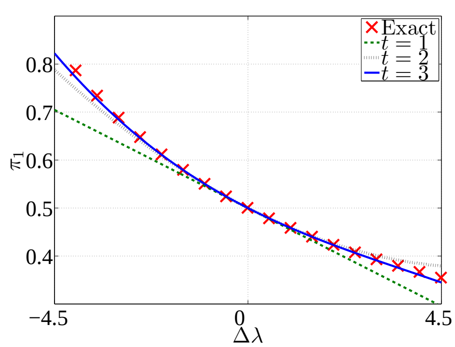


Figure 1. Polynomial approximations to model π_1.

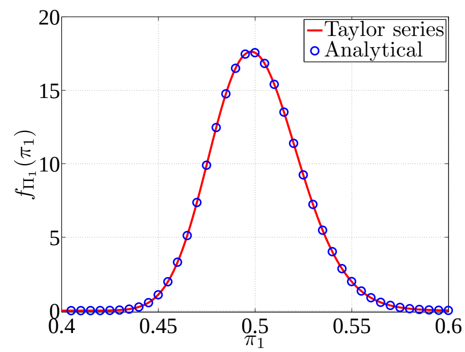


Figure 2. Taylor series and exact analytical results compared.

B. *Multiple-Parameter Case*

In this section, we consider the case where the generator matrix is a function of m parameters, $\theta_1, \theta_2, \ldots, \theta_m$, described by random variables, $\Theta_1, \Theta_2, \ldots, \Theta_m$. We assume that the Θ_j's are independent, and that the pdfs $f_{\Theta_j}(\theta_j)$, $j = 1, 2, \ldots, m$ are known.

1) Probability density function of Π_i: To derive the pdf of Π_i, we propose a method that builds upon the single-parameter case. First, we pick a parameter, say Θ_1, and seek the Taylor series expansion of Π_i around the mean of Θ_1, m_{Θ_1}, with the other parameters fixed. Along these lines, express Θ as

$$\Theta = m_\Theta + \Delta\Theta, \tag{35}$$

where $m_\Theta = [m_{\Theta_1},\, \theta_2, \ldots,\, \theta_m]$, and $\Delta\Theta = [\Delta\Theta_1,\, 0, \ldots,\, 0]$. We can expand $\pi_i(\Theta)$ around m_Θ using a Taylor series expansion as follows

$$\Pi_i = \pi_i(\Theta) = \pi_i(m_\Theta + \Delta\Theta) = \pi_i(m_\Theta) + \sum_{k=1}^{\infty} \frac{b_{ki}}{k!} \Delta\Theta_1^k. \tag{36}$$

The k-order Taylor series coefficient, b_{ki}, is given by:

$$b_{ki} \;=\; \left.\frac{\partial^k \pi_i(\theta)}{\partial \theta_1^k}\right|_{\theta=m_\Theta} = k!\,(-1)^k\,\pi(\theta)\left(\frac{\partial\Lambda(\theta)}{\partial\theta_1}\Lambda^{\#}\right)^k e_i^T\Bigg|_{\theta=m_\Theta}, \tag{37}$$

where $e_i \in \mathbb{R}^{n+1}$ is a row vector with 1 as the i entry and zero otherwise. We then express $\Pi_i = p_i(\Delta\Theta_1)$, where p_i is a polynomial function with real coefficients obtained by truncating the Taylor series in (36) at the t term:

$$\Pi_i = p_i(\Delta\Theta_1) = \pi_i(m_\Theta) + \sum_{k=1}^{t} \frac{b_{ki}}{k!} \Delta\Theta_1^k. \tag{38}$$

Analogous to (21), we can derive the conditional pdf

$$f_{\Pi_i|\Theta_2,\ldots,\Theta_m}\left(\pi_i|\theta_2,\ldots,\theta_m\right) = \sum_{j=1}^{r} \frac{f_{\Delta\Theta_1}(\Delta\theta_{1,j})}{|p_i'(\Delta\theta_{1,j})|}, \tag{39}$$

where $\Delta\theta_{1,1},\, \Delta\theta_{1,2}, \ldots \Delta\theta_{1,r}$ are the $r \leq t$ real roots of $\pi_i = p_i(\Delta\theta_1)$ and[4]

$$p_i'(\Delta\theta_{1,j}) = \left.\frac{dp_i(\Delta\theta_1)}{d\Delta\theta_1}\right|_{\Delta\theta_1=\Delta\theta_{1,j}} = \sum_{k=1}^{t} \frac{b_{ki}}{(k-1)!} \Delta\theta_{1,j}^{k-1}. \tag{40}$$

The derivation of (39) is provided in the Appendix. Applying the total probability theorem, and acknowledging the independence of $\Theta_2, \ldots, \Theta_m$, it follows that

$$f_{\Pi_i}\left(\pi_i\right) = \int\limits_{\theta_2} \ldots \int\limits_{\theta_m} f_{\Pi_i|\Theta_2,\ldots,\Theta_m}\left(\pi_i|\theta_2,\ldots,\theta_m\right) f_{\Theta_m}(\theta_m) d\theta_m \ldots f_{\Theta_2}(\theta_2) d\theta_2. \tag{41}$$

Remark 1. In the development above, the assumption of parameter independence is made from a modeling perspective. The proposed method is still mathematically tractable if the model

[4]Once the other parameters are fixed, p_i is a function of a single parameter $\Delta\theta_1$.

parameters are dependent, and their joint distribution is known. In particular, the conditional pdf $f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i|\theta_2,\ldots,\theta_m)$ in this case is given by:

$$f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i|\theta_2,\ldots,\theta_m) = \sum_{j=1}^{r} \frac{f_{\Delta\Theta_1|\Theta_2,\ldots,\Theta_m}(\Delta\theta_{1,j}|\theta_2,\ldots,\theta_m)}{|p_i'(\Delta\theta_{1,j})|}, \tag{42}$$

where $\Delta\theta_{1,1}, \Delta\theta_{1,2}, \ldots \Delta\theta_{1,r}$ are the $r \leq t$ real roots of $\pi_i = p_i(\Delta\theta_1)$. Appendix C includes a short note on the computation of $f_{\Delta\Theta_1|\Theta_2,\ldots,\Theta_m}(\Delta\theta_{1,j}|\theta_2,\ldots,\theta_m)$ from the joint distribution of the model parameters, and the subsequent derivation of $f_{\Pi_i}(\pi_i)$ from the total probability theorem.

2) Computer implementation: Algorithm 2 provides the pseudocode for computer implementation of the method outlined in (35)-(41) to compute $f_{\Pi_i}(\pi_i)$, $i = 0, 1, \ldots, n$ given $f_{\Theta_j}(\theta_j)$, $j = 1, 2, \ldots, m$. The vectors $\bar{\theta}_j = \left[\theta_j^{start} : d\theta_j : \theta_j^{end}\right]$, $j = 2, \ldots, m$ are defined so that each vector spans several standard deviations on both sides of m_{Θ_j}, the mean of Θ_j. The nested for loops ensure that the conditional pdf in (39) is evaluated point wise for the entries in $\bar{\theta}_j$. The QR factorization of the generator matrix is evaluated for every[5] $\hat{\theta} = \left[m_{\Theta_1}, \hat{\theta}_2, \ldots, \hat{\theta}_m\right]$, where $\hat{\theta}_j$, $j = 1, 2, \ldots, m$, denotes an entry of the vector $\bar{\theta}_j$. Next $\pi_i(\hat{\theta})$ is obtained from (12) by normalizing the last column of Q, the group inverse $\Lambda^{\#}$ is obtained from (13), and the Taylor series coefficients, b_{ki}, $k = 1, 2, \ldots, t$ are computed using (37). The r real roots of the equation $\hat{\pi}_i = p_i(\Delta\theta_1)$ are computed and the conditional $f_{\Pi_i|\Theta_2,\ldots,\Theta_m}\left(\hat{\pi}_i|\hat{\theta}_2,\ldots,\hat{\theta}_m\right)$ follows from (21)-(22). The integrals at the end of each nested for loop can be implemented using some numerical integration scheme, e.g., the trapezoidal method.

3) Expected Value and Variance of Π_i: To derive an expression for the expectation and variance of Π_i, consider the multiple-variable version of the Taylor series expansion

$$\Pi_i = \pi_i(\Theta) = \pi_i(m_\Theta + \Delta\Theta) = \pi_i(m_\Theta) + \sum_{k_1=1}^{\infty} \cdots \sum_{k_m=1}^{\infty} \frac{\Delta\Theta_1^{k_1} \ldots \Delta\Theta_m^{k_m}}{k_1! \ldots k_m!} \cdot \left. \frac{\partial^{k_1+\ldots k_m} \pi_i(\theta)}{\partial\theta_1^{k_1} \ldots \partial\theta_m^{k_m}} \right|_{\theta=m_\Theta}, \tag{43}$$

[5]Recall that in the pseudocode, we use the variable $\hat{\theta}_j$ to denote an entry in the vector $\bar{\theta}_j$, i.e., $\hat{\theta}_j = \bar{\theta}_j(l)$, for some l.

Algorithm 2 Computation of $f_{\Pi_i}(\pi_i)$ for the multi-parameter case

define $\bar{\pi}_i = [0 : d\pi_i : 1]$, $\bar{\theta}_2 = \left[\theta_2^{start} : d\theta_2 : \theta_2^{end}\right]$, ..., $\bar{\theta}_m = \left[\theta_m^{start} : d\theta_m : \theta_m^{end}\right]$
define Taylor series order t
compute $\frac{\partial \Lambda}{\partial \theta_j}$, $j = 2, \ldots, m$
for $\hat{\pi}_i = 0 : d\pi_i : 1$ **do**
  for $\hat{\theta}_2 = \theta_2^{start} : d\theta_2 : \theta_2^{end}$ **do**
    $\vdots$
    for $\hat{\theta}_m = \theta_m^{start} : d\theta_m : \theta_m^{end}$ **do**
      compute $QR = \Lambda(\hat{\theta})$, where $\hat{\theta} = \left[m_{\Theta_1},\, \hat{\theta}_2, \ldots,\, \hat{\theta}_m\right]$
      compute $\pi_i(\hat{\theta})$ from (12), $\Lambda^{\#}$ from (13), b_{ki}, $k = 1,\, 2,\, \ldots,\, t$ from (37)
      compute real roots of $\pi_i(\hat{\theta}) - \hat{\pi}_i + \sum_{k=1}^{t} \frac{b_{ki}}{k!} \Delta\theta_1^k = 0$, denote them by $\Delta\theta_{1,j}$, $j = 1,\, \ldots,\, r$
      for $j = 1$ to r **do**
        compute $f_{\Delta\Theta_1}(\Delta\theta_{1,j})$, and $p_i'(\Delta\theta_{1,j}) = \sum_{k=1}^{t} \frac{b_{ki}}{(k-1)!} \Delta\theta_{1,j}^{k-1}$
      end for
      compute $f_{\Pi_i|\Theta_2,\ldots,\Theta_m}\left(\hat{\pi}_i|\hat{\theta}_2, \ldots, \hat{\theta}_m\right) = \sum_{j=1}^{r} \frac{f_{\Delta\Theta_1}(\Delta\theta_{1,j})}{\left|p_i'(\Delta\theta_{1,j})\right|}$
    end for
    compute $f_{\Pi_i|\Theta_2,\ldots,\Theta_{m-1}}\left(\hat{\pi}_i|\hat{\theta}_2, \ldots, \hat{\theta}_{m-1}\right) = \int_{\theta_m} f_{\Pi_i|\Theta_2,\ldots,\Theta_m}\left(\hat{\pi}_i|\hat{\theta}_2, \ldots, \hat{\theta}_m\right) f_{\Theta_m}(\hat{\theta}_m) d\theta_m$
    $\vdots$
  end for
  compute $f_{\Pi_i}(\hat{\pi}_i) = \int_{\theta_2} f_{\Pi_i|\Theta_2}\left(\hat{\pi}_i|\hat{\theta}_2\right) f_{\Theta_2}(\hat{\theta}_2) d\theta_2$
end for

where $m_\Theta = [m_{\Theta_1},\, \mu_{\Theta_2}, \ldots,\, \mu_{\Theta_m}]$. While closed-form expressions for the partial derivatives $\partial^k \pi_i(\theta)/\partial\theta^k$ are available (see Theorem 1 in the Appendix), derivation of analytical expressions for the mixed partial derivatives of the form $\partial^{k_1+\ldots k_m}\pi_i(\theta)/\partial\theta_1^{k_1}\ldots\partial\theta_m^{k_m}$ is the focus of ongoing research. Therefore, we will focus on lower-order Taylor series expansions to approximate the expectation and variance of Π_i. Let us consider a second-order expansion for $\pi_i(\Theta)$:

$$\Pi_i \approx \pi_i(m_\Theta) + \Delta\Theta\; \nabla\pi_i(\theta)^T\Big|_{\theta=m_\Theta} + \frac{1}{2}\Delta\Theta\; \nabla^2\pi_i(\theta)\Big|_{\theta=m_\Theta} \Delta\Theta^T, \tag{44}$$

where $m_\Theta = [m_{\Theta_1},\, \mu_{\Theta_2}, \ldots,\, \mu_{\Theta_m}]$, and the gradient $\nabla\pi_i(\theta)$, and Hessian $\nabla^2\pi_i(\theta)$, are given by

$$\nabla\pi_i(\theta) = \left[\frac{\partial\pi_i(\theta)}{\partial\theta_1},\, \frac{\partial\pi_i(\theta)}{\partial\theta_2}, \ldots,\, \frac{\partial\pi_i(\theta)}{\partial\theta_m}\right], \tag{45}$$

$$\nabla^2 \pi_i(\theta) = \begin{bmatrix} \frac{\partial^2 \pi_i(\theta)}{\partial \theta_1^2} & \frac{\partial^2 \pi_i(\theta)}{\partial \theta_1 \partial \theta_2} & \cdots & \frac{\partial^2 \pi_i(\theta)}{\partial \theta_1 \partial \theta_m} \\ \frac{\partial^2 \pi_i(\theta)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \pi_i(\theta)}{\partial \theta_2^2} & \cdots & \frac{\partial^2 \pi_i(\theta)}{\partial \theta_2 \partial \theta_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi_i(\theta)}{\partial \theta_m \partial \theta_1} & \frac{\partial^2 \pi_i(\theta)}{\partial \theta_m \partial \theta_2} & \cdots & \frac{\partial^2 \pi_i(\theta)}{\partial \theta_m^2} \end{bmatrix}. \tag{46}$$

Substituting the gradient and Hessian in (44) and taking into account: i) the independence of the Θ_j's and, ii) the fact that $\mathrm{E}[\Delta\Theta_j] = 0 \, \forall j = 1, 2, \ldots, m$, the expected value of Π_i is given by

$$\begin{aligned} m_{\Pi_i} &= \pi_i(m_\Theta) + \frac{1}{2}\sum_{k=1}^{m} \mathrm{E}[\Delta\Theta_k^2] \cdot \left.\frac{\partial^2 \pi_i(\theta)}{\partial \theta_k^2}\right|_{\theta=m_\Theta} + \sum_{j=1}^{m} \sum_{k=1,\, k\neq j}^{m} \mathrm{E}[\Delta\Theta_j] \cdot \mathrm{E}[\Delta\Theta_k] \cdot \left.\frac{\partial \pi_i^2(\theta)}{\partial \theta_j \partial \theta_k}\right|_{\theta=m_\Theta} \\ &= \pi_i(m_\Theta) + \frac{1}{2}\sum_{k=1}^{m} \mathrm{E}[\Delta\Theta_k^2] \cdot \left.\frac{\partial^2 \pi_i(\theta)}{\partial \theta_k^2}\right|_{\theta=m_\Theta}. \end{aligned} \tag{47}$$

Similarly, assuming a first-order expansion for $\pi_i(\Theta)$,

$$\Pi_i \approx \pi_i(m_\Theta) + \Delta\Theta \, \nabla\pi_i(\theta)^T\big|_{\theta=m_\Theta}, \tag{48}$$

the variance of Π_i is given by

$$\sigma_{\Pi_i}^2 = \sum_{k=1}^{m} \mathrm{Var}[\Delta\Theta_k] \left.\left(\frac{\partial \pi_i(\theta)}{\partial \theta_k}\right)^2\right|_{\theta=m_\Theta}. \tag{49}$$

C. Uncertainty in Markov Reward Models

In this section, we show how the pdfs of the reward $\Xi = \Pi\rho^T$, and accumulated reward $\Gamma = \Xi \cdot \tau$—denoted by $f_\Xi(\xi)$ and $f_\Gamma(\gamma)$, respectively—can be computed for the multiple-parameter case. We also propose closed-form approximations for the expectation and variance of Ξ and Γ.

1) Probability density function of Ξ and Γ: To derive the pdf of Ξ, we follow a procedure similar to the one outlined in Section III-B. First, we pick a parameter, say Θ_1, and seek the Taylor series expansion of ξ around the mean of Θ_1, with the other parameters fixed. As before, splitting Θ as

$$\Theta = m_\Theta + \Delta\Theta, \tag{50}$$

where $m_\Theta = [m_{\Theta_1}, \theta_2, \ldots, \theta_m]$ and $\Delta\Theta = [\Delta\Theta_1, 0, \ldots, 0]$, we can express

$$\Xi \;=\; \xi(\Theta) = \xi(m_\Theta + \Delta\Theta) = \xi(m_\Theta) + \sum_{k=1}^{\infty} \frac{c_k}{k!} \Delta\Theta_1^k = \pi(m_\Theta)\rho^T + \sum_{k=1}^{\infty} \frac{c_k}{k!} \Delta\Theta_1^k. \tag{51}$$

The k-order Taylor series coefficient, c_k, is given by

$$c_k \;=\; \left.\frac{\partial^k \xi(\theta)}{\partial \theta_1^k}\right|_{\theta=m_\Theta} = \left.\frac{\partial^k \pi(\theta)}{\partial \theta_1^k}\rho^T\right|_{\theta=m_\Theta} = k!\,(-1)^k\,\pi(\theta)\left(\frac{\partial \Lambda(\theta)}{\partial \theta_1}\Lambda^{\#}\right)^k \rho^T\Bigg|_{\theta=m_\Theta}. \tag{52}$$

We then express $\Xi = x(\Delta\Theta_1)$, where x is a polynomial function with real coefficients obtained by truncating the Taylor series in (51) at the t term:

$$\Xi = x(\Delta\Theta_1) = \pi(m_\Theta)\rho^T + \sum_{k=1}^{t} \frac{c_k}{k!} \Delta\Theta_1^k. \tag{53}$$

Then, analogous to (39), we get

$$f_{\Xi|\Theta_2,\ldots,\Theta_m}\left(\xi|\theta_2,\ldots,\theta_m\right) = \sum_{j=1}^{r} \frac{f_{\Delta\Theta_1}(\Delta\theta_{1,j})}{|x'(\Delta\theta_{1,j})|}, \tag{54}$$

where $\Delta\theta_{1,1},\, \Delta\theta_{1,2}, \ldots \Delta\theta_{1,r}$ are the $r \le t$ roots of $\xi = x(\Delta\theta_1)$. Applying the total probability theorem, and acknowledging the independence of $\Theta_2, \ldots, \Theta_m$, it follows that

$$f_\Xi\left(\xi\right) = \int\limits_{\theta_2} \cdots \int\limits_{\theta_m} f_{\Xi|\Theta_2,\ldots,\Theta_m}\left(\xi|\theta_2,\ldots,\theta_m\right) f_{\Theta_m}(\theta_m) d\theta_m \ldots f_{\Theta_2}(\theta_2) d\theta_2. \tag{55}$$

From the definition of $\Gamma = \Xi \cdot \tau$, it follows that

$$f_\Gamma(\gamma) = \frac{f_\Xi(\gamma/\tau)}{\tau}. \tag{56}$$

2) Computer Implementation: Algorithm 3 provides the pseudocode for computer implementation of the method outlined in (50)-(56) to compute $f_\Xi(\xi)$ and $f_\Gamma(\gamma)$ given $f_{\Theta_j}(\theta_j)$, $j = 1,\, 2, \ldots,\, m$. The pseudocode follows along similar lines to that in Algorithm 2. Note that the vectors $\bar{\xi} = [0 : d\xi : \|\rho\|_1]$ and $\bar{\gamma} = [0 : d\gamma : \tau \cdot \|\rho\|_1]$ are formulated based on the one-norm of ρ, since $\xi = \pi\rho^T$, $\gamma = \xi \cdot \tau = \pi \cdot \rho^T \cdot \tau$ and $0 \le \pi_i \le 1$, $\forall i = 0,\, 1, \ldots,\, n$.

Algorithm 3 Computation of $f_{\Xi}(\xi)$ and $f_{\Gamma}(\gamma)$ for the multiple-parameter case

define $\rho = [\rho_0, \rho_1, \ldots, \rho_n]$, τ, $\bar{\xi} = [0 : d\xi : \|\rho\|_1]$, $\bar{\gamma} = [0 : d\gamma : \tau \cdot \|\rho\|_1]$, $\bar{\theta}_2 = \left[\theta_2^{start} : d\theta_2 : \theta_2^{end}\right]$,..., $\bar{\theta}_m = \left[\theta_m^{start} : d\theta_m : \theta_m^{end}\right]$
define Taylor series order t
compute $\frac{\partial \Lambda}{\partial \theta_j}$, $j = 2, \ldots, m$
for $\hat{\xi} = 0 : d\xi : \|\rho\|_1$ **do**
  for $\hat{\theta}_2 = \theta_2^{start} : d\theta_2 : \theta_2^{end}$ **do**
    $\vdots$
    for $\hat{\theta}_m = \theta_m^{start} : d\theta_m : \theta_m^{end}$ **do**
      compute $QR = \Lambda(\hat{\theta})$ where $\hat{\theta} = \left[m_{\Theta_1}, \hat{\theta}_2, \ldots, \hat{\theta}_m\right]$
      compute $\xi(\hat{\theta}) = \pi(\hat{\theta}) \cdot \rho^T$ using (12), $\Lambda^{\#}$ from (13), c_k, $k = 1, 2, \ldots, t$ from (52)
      compute real roots of $\xi(\hat{\theta}) - \hat{\xi} + \sum_{k=1}^{t} \frac{c_k}{k!} \Delta\theta_1^k = 0$, denote them by $\Delta\theta_{1,j}$, $j = 1, \ldots, r$
      for $j = 1$ to r **do**
        compute $f_{\Delta\Theta_1}(\Delta\theta_{1,j})$, and $x'(\Delta\theta_{1,j}) = \sum_{k=1}^{t} \frac{c_k}{(k-1)!} \Delta\theta_{1,j}^{k-1}$
      end for
      compute $f_{\Xi|\Theta_2,\ldots,\Theta_m}\left(\hat{\xi}|\hat{\theta}_2, \ldots, \hat{\theta}_m\right) = \sum_{j=1}^{r} \frac{f_{\Delta\Theta_1}(\Delta\theta_{1,j})}{|x'(\Delta\theta_{1,j})|}$
    end for
    compute $f_{\Xi|\Theta_2,\ldots,\Theta_{m-1}}\left(\hat{\xi}|\hat{\theta}_2, \ldots, \hat{\theta}_{m-1}\right) = \int_{\theta_m} f_{\Xi|\Theta_2,\ldots,\Theta_m}\left(\hat{\xi}|\hat{\theta}_2, \ldots, \hat{\theta}_m\right) f_{\Theta_m}(\hat{\theta}_m) d\theta_m$
    $\vdots$
  end for
  compute $f_{\Xi}(\hat{\xi}) = \int_{\theta_2} f_{\Xi|\Theta_2}(\hat{\xi}|\hat{\theta}_2) f_{\Theta_2}(\hat{\theta}_2) d\theta_2$
end for
for $\hat{\gamma} = 0 : d\gamma : \tau \cdot \|\rho\|_1$ **do**
  $f_{\Gamma}(\hat{\gamma}) = \frac{f_{\Xi}(\hat{\gamma}/\tau)}{\tau}$
end for

3) Expected Value and Variance of Ξ and Γ: Similar to (47), assuming a second-order expansion for $\xi(\Theta)$, we can express the expected value of Ξ, denoted by m_{Ξ}, as follows

$$m_{\Xi} \equiv \mathrm{E}[\Xi] = \pi(m_{\Theta})\rho^T + \frac{1}{2} \sum_{k=1}^{m} \mathrm{E}[\Delta\Theta_k^2] \left. \frac{\partial^2 \pi(\theta)}{\partial \theta_k^2} \right|_{\theta = m_{\Theta}} \rho^T. \tag{57}$$

Additionally, similar to (49), assuming a first-order expansion for $\xi(\Theta)$, the variance of Ξ, denoted by σ_{Ξ}^2, is given by

$$\sigma_{\Xi}^2 \equiv \mathrm{Var}(\Xi) = \sum_{k=1}^{m} \mathrm{Var}\left(\Delta\Theta_k\right) \left. \left(\frac{\partial \pi(\theta)}{\partial \theta_k} \rho^T \right)^2 \right|_{\theta = m_{\Theta}} \tag{58}$$

From the definition of $\Gamma = \Xi \cdot \tau$, we can estimate the expected value and variance of Γ, (denoted by m_Γ and σ_Γ^2, respectively) as follows

$$m_\Gamma \equiv \mathrm{E}[\Gamma] = \mathrm{E}[\Xi \cdot \tau] = m_\Xi \cdot \tau, \tag{59}$$

$$\sigma_\Gamma^2 \equiv \mathrm{Var}(\Gamma) = \mathrm{Var}(\Xi \cdot \tau) = \sigma_\Xi^2 \cdot \tau^2. \tag{60}$$

IV. Case Studies

The first case study returns to the two-state Markov model discussed in Example 1. While the example explored a single uncertain parameter, in this case study, we consider the case where both parameters are uncertain. It is still fairly straightforward to derive an analytical expression for the pdfs of the stationary distribution, the reward, and hence the accumulated reward, because the steady-state probabilities are simple functions of the model parameters. The availability of an analytical solution allows us to validate the Taylor series approach. The second case study explores a two-component load-sharing system with common-cause failures [14]. In this case, it is not possible to derive the pdfs of the steady-state probabilities and the reward from the analytical expressions of the stationary distribution. Therefore the results from the Taylor series approach are compared with those obtained from repeated Monte Carlo simulations. In the final case study, we examine computer execution times for an $n+1$ state reward model for a system of n identical components, each with two operating modes.

In all the case studies that follow, we model the failure rates with normal distributions and repair rates with uniform distributions. This is based on the presumption that typically the mean and variance of the failure rate might be available from field data; however, due to the involvement of myriad human factors, only a range of repair times might be known. Also, note that failure and repair rates have units of per-unit time. To streamline the presentation, we omit the units in the following discussion.

A. Single Component with Two Operating States

Consider the component reliability model of Example 1. The generator matrix and stationary distribution of the chain are given by (27) and (28), respectively. We define a reward model by choosing a reward vector $\rho = [\rho_0, \, \rho_1]$, where ρ_0 and ρ_1 are constants that capture some notion of performance while in states 0 and 1, respectively. As described in (6), the long-term reward is given by $\xi = \rho_0 \pi_0 + \rho_1 \pi_1$, and as described in (7), the accumulated reward at time τ, $\gamma = \xi \cdot \tau$.

Since the failure and repair rates are not perfectly known, it is assumed that they are described by random variables L and M with (known) pdfs $f_L(\lambda)$ and $f_M(\mu)$, respectively. Further, it is assumed that L and M are independent. Consequently, the stationary distribution is described by random variables Π_0 and Π_1, and the reward and accumulated reward are described by random variables Ξ and Γ, respectively. Through random variable transformations, the following expressions for $f_{\Pi_0}(\pi_0)$, $f_{\Pi_1}(\pi_1)$, and $f_{\Xi}(\xi)$ can be derived from the closed-form expressions for π_0 and π_1 given in (28):

$$f_{\Pi_0}(\pi_0) = \int\limits_{\lambda} \frac{\lambda}{\pi_0^2} \cdot f_M\left(\frac{\lambda(1-\pi_0)}{\pi_0}\right) \cdot f_L(\lambda) d\lambda, \tag{61}$$

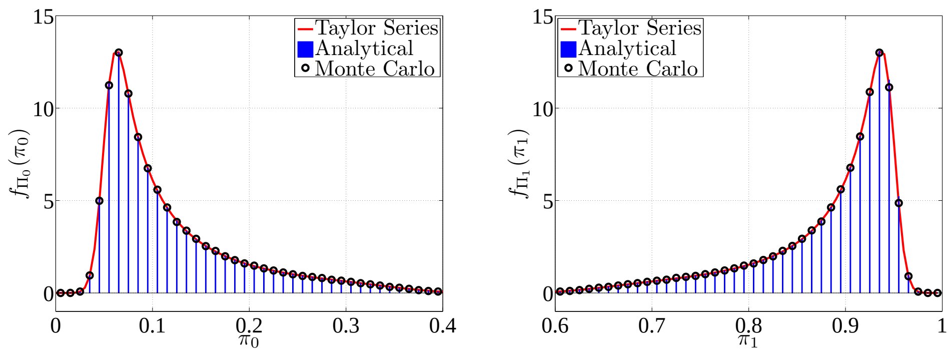
$$f_{\Pi_1}(\pi_1) = \int\limits_{\lambda} \frac{\lambda}{(1-\pi_1)^2} \cdot f_M\left(\frac{\lambda \pi_1}{(1-\pi_1)}\right) \cdot f_L(\lambda) d\lambda, \tag{62}$$

$$f_{\Xi}(\xi) = \int\limits_{\lambda} \left| \frac{\lambda\left(\rho_1 - \rho_0\right)}{\left(\rho_1 - \xi\right)^2} \right| \cdot f_M\left(\frac{\lambda\left(\xi - \rho_0\right)}{\left(\rho_1 - \xi\right)}\right) \cdot f_L(\lambda) d\lambda. \tag{63}$$

Recall that $f_\Gamma(\gamma)$ can be obtained from $f_{\Xi}(\xi)$ using (56).

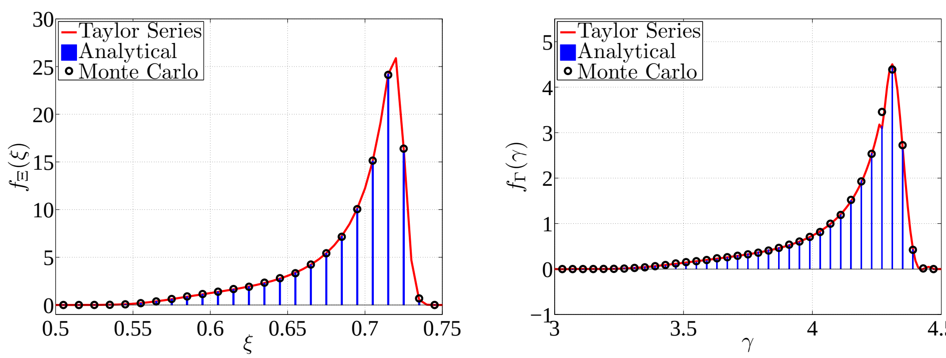
For illustration, let us consider that the failure rate is normally distributed and that the repair rate is uniformly distributed, i.e., $L \sim \mathcal{N}(m_L, \, \sigma_L^2)$, $M \sim \mathcal{U}(a_M, \, b_M)$. Figures 3, 4 depict the pdfs $f_{\Pi_0}(\pi_0)$, $f_{\Pi_1}(\pi_1)$, $f_{\Xi}(\xi)$, and $f_\Gamma(\gamma)$ computed: i) numerically using a third-order Taylor series expansion with the methods outlined in Sections III-B2 and III-C2, ii) analytically through (61), (62), (63), and iii) numerically from a 1,000,000-sample Monte Carlo simulation performed as follows. We first sample the distribution of the random vector Θ that describes the values that the model parameters can take. For each sample θ, we obtain the corresponding generator matrix $\Lambda(\theta)$ by substituting for the corresponding values. Then, by using a QR factorization of $\Lambda(\theta)$, we obtain the stationary distribution of the chain $\pi(\theta)$ without having to solve the Chapman-Kolmogorov equations (for the specific $\Lambda(\theta)$ as $t \to \infty$). The simulation parameters are: $m_L = 0.55$, $\sigma_L^2 = 0.1^2$, $a_M = 1$, $b_M = 10$, $\rho = [\rho_0, \rho_1] = [0.25, 0.75]$, and $\tau = 6$. The results indicate that the pdfs computed via the Taylor series method accurately match the exact analytical results and those from Monte Carlo simulations.

Table I lists the analytically computed expectations and variances for Π_0, Π_1, Ξ, and Γ for two sets of parameter distributions: $L \sim \mathcal{N}(0.55, 0.1^2)$, $M \sim \mathcal{U}(1, 10)$, and $L \sim \mathcal{N}(0.55, 0.1^2)$, $M \sim \mathcal{U}(100, 109)$. Recall that the analytical expressions for the expectation and variance are based on lower-order approximations derived in Sections III-B3 and III-C3. For comparison, the expectations and variances computed numerically from their pdfs—derived using the third-order Taylor series expansion—are also computed. The expectations computed analytically match those computed numerically in both cases. However, the analytically computed variance matches the exact numerical result only when the mean repair rate is several orders of magnitude larger than the mean failure rate. Note that since the expectation and variance are computed assuming second- and first-order truncations of the Taylor-series expansion, there might be an error introduced in the computed values if higher-order terms are dominant. For the examples we explore in the case studies, the higher-order terms are negligible if the mean repair rate is several orders of magnitude higher than the mean failure rate—consequently, the analytical results match the numerical values better in these cases. While the results may be inaccurate, the analytical expressions can be evaluated with minimum effort, and thus serve useful for back-of-the-envelope calculations. On the other hand, the pdfs computed following the Taylor-series method are accurate (even if the analytically computed moments are not accurate). This is because the method proposed to obtain the pdfs of the reliability indices does not constrain the order of the Taylor-series expansion. We obtain very accurate estimates for the mean and variance of the indices from the computed pdfs—at the expense of computation time.



(a) (b)

Figure 3. $f_{\Pi_0}(\pi_0)$, $f_{\Pi_1}(\pi_1)$ for $L \sim \mathcal{N}(0.55,\, 0.1^2)$ and $M \sim \mathcal{U}(1,\, 10)$.



(a) (b)

Figure 4. $f_{\Xi}(\xi)$, $f_{\Gamma}(\gamma)$ for $L \sim \mathcal{N}(0.55,\, 0.1^2)$, $M \sim \mathcal{U}(1,\, 10)$, $\rho = [0.25,\, 0.75]$, and $\tau = 6$.

B. Two-Component Load-Sharing System with Common-Cause Failures

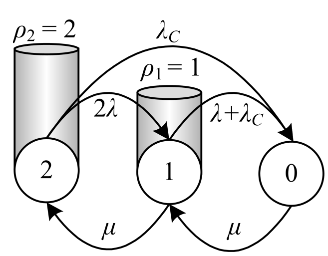


Figure 5. State-transition diagram for load-sharing system with common-cause failures.

Table I
ANALYTICAL AND NUMERICAL EXPECTATIONS AND VARIANCES COMPARED FOR CASE A

Case	R.v.	m: Analytical	m: Numerical	σ^2: Analytical	σ^2: Numerical
$L \sim \mathcal{N}(0.55,\, 0.1^2)$ $M \sim \mathcal{U}(1,\, 10)$	Π_0	0.1019	0.1167	0.0012	0.0053
	Π_1	0.8981	0.8833	0.0012	0.0053
	Ξ	0.6991	0.6916	3.1151e-4	0.0013
	Γ	4.1943	4.1328	0.0112	0.0477
$L \sim \mathcal{N}(0.55,\, 0.1^2)$ $M \sim \mathcal{U}(100,\, 109)$	Π_0	0.0052	0.0052	9.1312e-7	9.1501e-7
	Π_1	0.9948	0.9948	9.1312e-7	9.1501e-7
	Ξ	0.7474	0.7474	2.2828e-7	2.2875e-7
	Γ	4.4843	4.4954	8.2181e-6	8.1940e-6

This example, adapted from [14], explores a system composed of two identical components that share a common load. The component failure rate is denoted by λ, and the repair rate is denoted by μ. In addition, the system is susceptible to common-cause failures which cause all operational components to fail at the same time. The common-cause failure rate is denoted by λ_C. The state transition diagram for this system is depicted in Fig. 5. Both components are operational in state 2, a single component is operational in state 1, and in state 0, both components have failed. Repairs restore the operation of one component at a time. From the state-transition diagram in Fig. 5, the generator matrix can be derived as

$$\Lambda = \begin{bmatrix} -\mu & \mu & 0 \\ \lambda + \lambda_C & -(\lambda + \lambda_C + \mu) & \mu \\ \lambda_C & 2\lambda & -(2\lambda + \lambda_C) \end{bmatrix}. \tag{64}$$

Denote the stationary distribution of the chain by $\pi = [\pi_0,\, \pi_1,\, \pi_2]$. Solving (4), we obtain [14]

$$\pi_0 = \frac{(\lambda + \lambda_C)(2\lambda + \lambda_C) + \lambda_C \mu}{(\lambda + \lambda_C + \mu)(2\lambda + \lambda_C) + \lambda_C \mu + \mu^2}, \tag{65}$$

$$\pi_1 = \frac{(2\lambda + \lambda_C)\mu}{(\lambda + \lambda_C + \mu)(2\lambda + \lambda_C) + \lambda_C \mu + \mu^2}, \tag{66}$$

$$\pi_2 = \frac{\mu^2}{(\lambda + \lambda_C + \mu)(2\lambda + \lambda_C) + \lambda_C \mu + \mu^2}. \tag{67}$$

Notice how involved the analytical closed-form expressions are even for this simple system. Consider that the performance of the system is proportional to the number of operational components. Then, we can define a reward model for this system by choosing $\rho = [\rho_0,\, \rho_1,\, \rho_2] = [0,\, 1,\, 2]$. The long-term reward is given by $\xi = \pi \cdot \rho^T = \pi_1 + 2\pi_2$.

Suppose the failure rate, repair rate, and common-cause failure rate are described by random variables L, M, and L_C with (known) pdfs $f_L(\lambda)$, $f_M(\mu)$, and $f_{L_C}(\lambda_C)$, respectively. Additionally, it is assumed that L, M, and L_C are independent. Consequently, the components of the stationary distribution $\Pi = [\Pi_0\,, \Pi_1,\, \Pi_2]$ are random variables with distributions $f_{\Pi_0}(\pi_0)$, $f_{\Pi_1}(\pi_1)$, and $f_{\Pi_2}(\pi_2)$. Similarly, the reward, $\Xi = \Pi \cdot \rho^T = \Pi_1 + 2\Pi_2$ is a random variable with distribution $f_\Xi(\xi)$. Unlike the two-state example explored in Section IV-A, it is clear from the expressions of the steady-state probabilities that closed-form expressions for the pdfs cannot be obtained easily. Therefore, we recourse to the Taylor series approach to derive the pdfs of the steady-state probabilities and the reward.

Let us consider $L \sim \mathcal{N}(0.5,\, 0.1^2)$, $L_C \sim \mathcal{N}(0.05,\, 0.01^2)$, and $M \sim \mathcal{U}(1,\, 10)$. Figure 6 depicts the pdfs $f_{\Pi_0}(\pi_0)$, $f_{\Pi_1}(\pi_1)$, $f_{\Pi_2}(\pi_2)$, and $f_\Xi(\xi)$, all computed using a third-order Taylor series expansion with the methods outlined in Sections III-B2 and III-C2. Additionally, results from a 1,000,000-sample Monte Carlo simulation are also shown. The figures indicate that the pdfs computed via the Taylor series method accurately match those obtained through Monte Carlo simulations.

Table I lists the analytically computed expectations and variances for Π_0, Π_1, Π_2, and Ξ for two sets of parameter distributions: $L \sim \mathcal{N}(0.5, 0.1^2)$, $L_C \sim \mathcal{N}(0.05,\, 0.01^2)$, $M \sim \mathcal{U}(1,\, 10)$, and $L \sim \mathcal{N}(\text{1.6e-4},\, (\text{25e-6})^2)$, $L_C \sim \mathcal{N}(\text{2e-5},\, (\text{5e-6})^2)$, $M \sim \mathcal{U}(0.1, 0.15)$. Recall that the analytical expressions for the expectation and variance are based on lower-order approximations derived in Sections III-B3 and III-C3. For comparison, the expectations and variances computed numerically from their pdfs—derived by the third-order Taylor series approach—are also computed. As before, while the expectations computed analytically match those computed numerically in both cases, the analytically computed variance matches the exact numerical result only when the mean repair rate is several orders of magnitude larger than the mean failure rate.

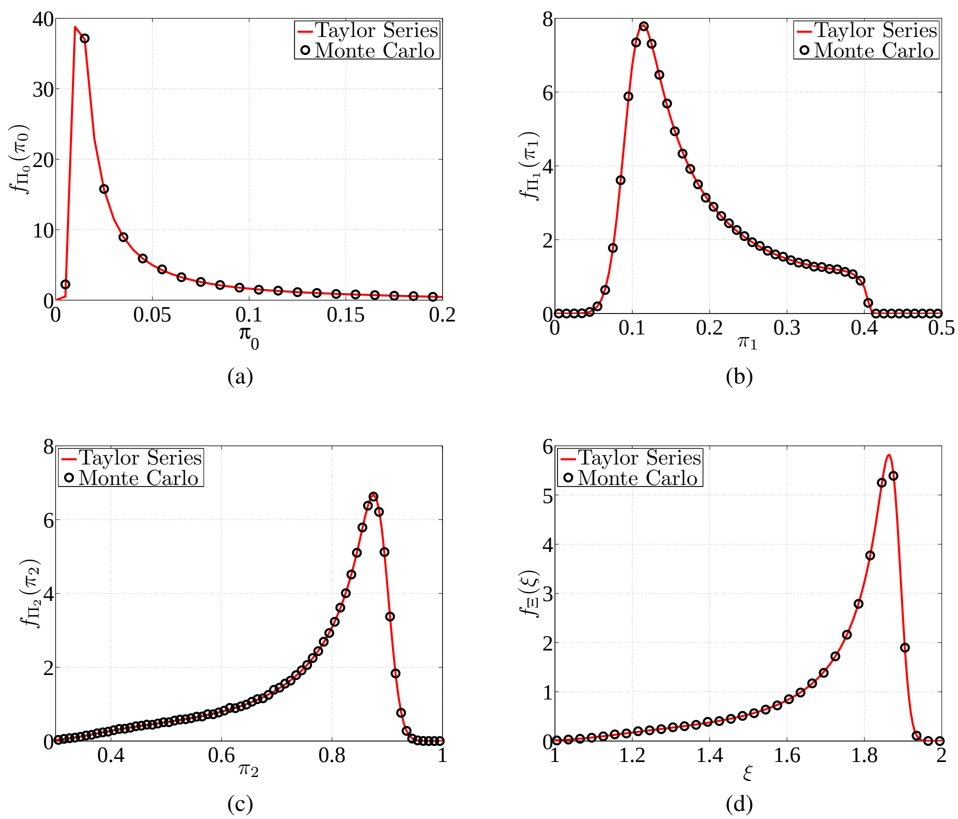


Figure 6. $f_{\Pi_0}(\pi_0)$, $f_{\Pi_1}(\pi_1)$, $f_{\Pi_2}(\pi_2)$, and $f_{\Xi}(\xi)$ for $L \sim \mathcal{N}(0.5,\, 0.1^2)$, $L_C \sim \mathcal{N}(0.05,\, 0.01^2)$, $M \sim \mathcal{U}(1,\, 10)$, $\rho = [0,\, 1,\, 2]$.

C. System of n Components

The final case study compares the execution time, t_e, of the proposed Taylor series method with Monte Carlo simulations for a system of n identical components, each with two operating modes (functioning/failed). The state-transition diagram that models the reliability of this system is depicted in Fig. 7. The component failure rate is denoted by λ, and the repair rate is denoted by μ. Repairs restore the operation of all failed components simultaneously. The performance of the system is proportional to the number of operational components. A reward model for this system is formulated by choosing $\rho = [\rho_0,\, \rho_1,\, \dots \rho_i, \dots,\, \rho_n] = \left[0,\, \frac{1}{n},\, \dots \frac{i}{n}, \dots,\, 1\right]$. The long-term

Table II
ANALYTICAL AND NUMERICAL EXPECTATIONS AND VARIANCES COMPARED FOR CASE B

Case	R.v.	m: Analytical	m: Numerical	σ^2: Analytical	σ^2: Numerical
$L \sim \mathcal{N}(0.5, 0.1^2)$ $L_C \sim \mathcal{N}(0.05,\ 0.01^2)$ $M \sim \mathcal{U}(1,\ 10)$	Π_0	0.0330	0.0430	3.8745e-5	0.0024
	Π_1	0.1777	0.1822	6.6013e-4	0.0070
	Π_2	0.7892	0.7748	0.0010	0.0167
	Ξ	1.7562	1.7318	0.0014	0.0312
$L \sim \mathcal{N}(1.6\text{e-}4,\ (25\text{e-}6)^2)$ $L_C \sim \mathcal{N}(2\text{e-}5,\ (5\text{e-}6)^2)$ $M \sim \mathcal{U}(0.1, 0.15)$	Π_0	1.6579e-4	1.6683e-4	1.9755e-9	2.1305e-9
	Π_1	0.0027	0.0027	2.5731e-7	2.6977e-7
	Π_2	0.9971	0.9968	2.7533e-7	2.8889e-7
	Ξ	1.9969	1.9964	2.9730e-7	3.1215e-7

reward is given by $\xi = \pi\rho^T = \frac{1}{n}\pi_1 + \frac{2}{n}\pi_2 + \ldots + \frac{i}{n}\pi_i + \ldots + \pi_n$. Suppose the failure rate and repair rate are described by random variables $L \sim \mathcal{N}(0.55,\ 0.1^2)$ and $M \sim \mathcal{U}(1,\ 10)$, respectively. Consequently, the components of the stationary distribution $\Pi = [\Pi_0,\ \Pi_1,\ \ldots,\ \Pi_i, \ldots,\ \Pi_n]$ are random variables, and the reward, $\Xi = \Pi\rho^T$ is a random variable with pdf $f_\Xi(\xi)$.

This case study explores the impact of the number of samples in the Monte Carlo simulation, n_s, and the dimension of the state space, n, on the time to compute $f_\Xi(\xi)$ through: i) third-order Taylor series approach following the pseudocode outlined in Section III-C2, ii) Monte Carlo simulations involving repeated sampling from n_s-length random samples of the failure and repair-rate distributions. The experiment is performed on a PC with a 2.66 GHz Intel®

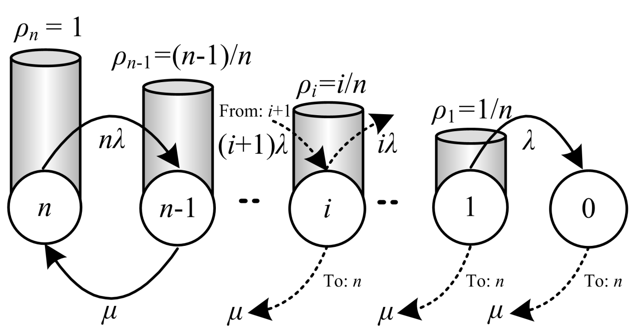


Figure 7. State-transition diagram for system of n components.

Core$^{\text{TM}}$2 Quad CPU processor with 4 GB memory in the MATLAB® environment. Figure 8 plots the percentage difference in the variance of Ξ, σ_Ξ, as a function of n_s for $n = 2$ for one experimental run. The result demonstrates the significance of n_s on the accuracy of Monte Carlo simulations. In the Monte Carlo simulation, for 100 runs (with 75,000 samples in each run), the mean percentage error is 0.58%. Figure 9 plots the execution time of the two methods as a function of n and n_s. In the experiment, n is increased from 2 to 20 in steps of 2, and n_s is increased from 65,000 to 75,000 in steps of 500. The Taylor series method execution time is lower than Monte Carlo simulations over a wide range of n_s (prominent for $n_s > 70,000$). For large models ($n > 20$) and a sufficiently large number of samples ($n_s > 75,000$), Fig. 9 clearly indicates that the proposed Taylor series method outperforms Monte Carlo simulation.

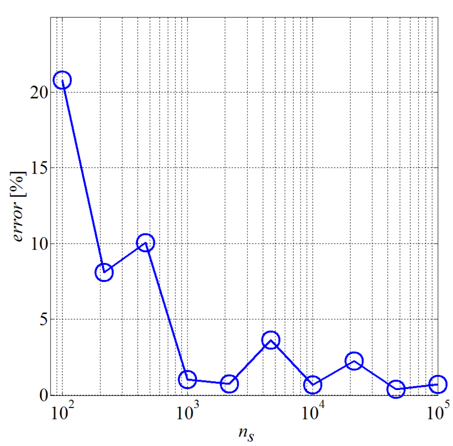


Figure 8. Percentage error in σ_Ξ as a function of n_s, $n = 2$.

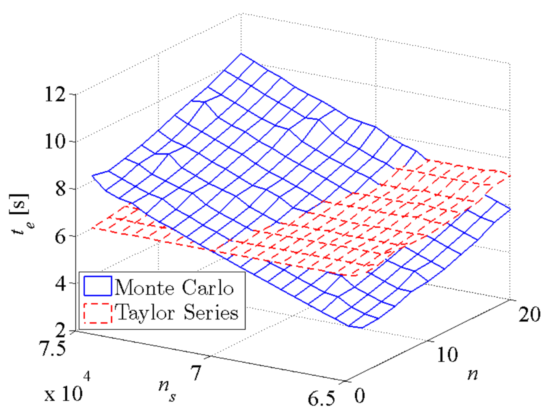


Figure 9. Execution time t_e, as a function of model order n, and number of samples n_s.

V. Concluding Remarks

A numerical method based on the Taylor series expansion of the Markov chain stationary distribution is proposed to propagate parametric uncertainty to reliability and performability indices in Markov reliability and reward models. The proposed method allows the computation of the probability distributions of these indices given distributions of the uncertain parameters. Closed-form approximations for the expectation and variance of the indices are also proposed.

The main advantage of the proposed framework is that only the generator matrix is required as the input. Additionally, for large models with few uncertain parameters, the proposed method demonstrates significantly lower execution time when compared to Monte Carlo simulations. As part of future work, we could further investigate the multivariate Taylor-series expansion. Comparing the efficiency and accuracy of the method proposed here to compute the sensitivities to that suggested in prior work (e.g., [11], [4] and the references therein), is another aspect that can be investigated.

APPENDIX

A. Derivation of stationary distribution sensitivity

Theorem 1. *The k-order sensitivity of the stationary distribution, $\pi(\theta)$ of an ergodic continuous time Markov chain (CTMC) described by (3) with respect to the i model parameter, θ_i, is given by*

$$\frac{\partial^k \pi(\theta)}{\partial \theta_i^k} = k!\,(-1)^k\,\pi(\theta)\left(\frac{\partial \Lambda}{\partial \theta_i}\Lambda^{\#}\right)^k, \tag{68}$$

where $\Lambda^{\#}$ is the group inverse of the generator matrix Λ.

Proof: Consider that the ergodic CTMC is associated with a discrete time Markov chain (DTMC) whose distribution is governed by

$$p[k+1] = p[k]P, \tag{69}$$

where $P = I + \delta\Lambda$ is a row-stochastic, irreducible, and primitive matrix (with an appropriate choice of δ). Define the matrix

$$A = I - P = -\delta\Lambda, \tag{70}$$

and denote the group inverse of A by $A^{\#}$. The stationary distribution of the DTMC satisfies $pA = 0$. If we consider linear perturbations, i.e., $\partial^k A/\partial\theta_i^k = 0, \forall k > 1$, differentiating the expression $pA = 0$ a total of k times yields

$$\frac{\partial^k p}{\partial \theta_i^k} A = -k \frac{\partial^{k-1} p}{\partial \theta_i^{k-1}} \frac{\partial A}{\partial \theta_i}. \tag{71}$$

Following along the lines of Theorem 3.2 in [19], since $dim\, N(A) = 1$ (the null space of a matrix A is denoted by $N(A)$), we can express

$$\frac{\partial^k p}{\partial \theta_i^k} = -k \frac{\partial^{k-1} p}{\partial \theta_i^{k-1}} \frac{\partial A}{\partial \theta_i} A^{\#} + \alpha p \text{ for some } \alpha. \tag{72}$$

We can determine α by noting that $pe^T = 1 \Rightarrow \partial^k p / \partial \theta_i^k e^T = 0$. Since $e^T \in N(A) = N(A^{\#})$,

$$\frac{\partial^k p}{\partial \theta_i^k} e^T = -k \frac{\partial^{k-1} p}{\partial \theta_i^{k-1}} \frac{\partial A}{\partial \theta_i} A^{\#} e^T + \alpha p e^T = \alpha p e^T = 0 \Rightarrow \alpha = 0. \tag{73}$$

Thus the k-order sensitivity of the stationary distribution of the DTMC to the i parameter is given by

$$\frac{\partial^k p}{\partial \theta_i^k} = -k \frac{\partial^{k-1} p}{\partial \theta_i^{k-1}} \frac{\partial A}{\partial \theta_i} A^{\#}. \tag{74}$$

Expressing $\partial^{k-1} p / \partial \theta_i^{k-1}$ as a function of $\partial^{k-2} p / \partial \theta_i^{k-2}$ and so on, we get

$$\begin{aligned} \frac{\partial^k p}{\partial \theta_i^k} &= k!(-1)^{k-1} \frac{\partial p}{\partial \theta_i} \left(\frac{\partial A}{\partial \theta_i} A^{\#} \right)^{k-1} \\ &= k!(-1)^k p(\theta) \left(\frac{\partial A}{\partial \theta_i} A^{\#} \right)^k, \end{aligned} \tag{75}$$

which follows from the result

$$\frac{\partial p}{\partial \theta_i} = -p(\theta) \frac{\partial A}{\partial \theta_i} A^{\#}, \tag{76}$$

derived in Theorem 3.2 in [19]. Now, consider that the group inverse of the CTMC generator matrix, Λ, denoted by $\Lambda^{\#}$, is given by

$$\Lambda^{\#} = -\delta A^{\#}, \tag{77}$$

which can be shown by noting that $\Lambda^{\#}$ satisfies the definition of the group inverse given in (10).

From (77) and (70),

$$\frac{\partial \Lambda(\theta)}{\partial \theta_i}\Lambda^{\#} = \left(-\delta^{-1}\frac{\partial A(\theta)}{\partial \theta_i}\right)\left(-\delta A^{\#}\right) = \frac{\partial A(\theta)}{\partial \theta_i}A^{\#}. \tag{78}$$

Since the stationary distributions of the CTMC and the DTMC match, from (75) and (78):

$$\frac{\partial^k \pi(\theta)}{\partial \theta_i^k} = k!\,(-1)^k\,\pi(\theta)\left(\frac{\partial \Lambda}{\partial \theta_i}\Lambda^{\#}\right)^k. \tag{79}$$

■

B. Derivation of result in (39)

The expression in (39) can be derived as follows[6]:

$$\begin{aligned}
&\Pr\left\{\pi_i \le \Pi_i \le \pi_i + \Delta(\pi_i)\,|\,\Theta_2 = \theta_2,\, \Theta_3 = \theta_3, \ldots,\, \Theta_m = \theta_m\right\} \\
= &\sum_{j\in\mathcal{J}^-} \Pr\left\{\Delta\theta_{1,j} + \Delta(\Delta\theta_{1,j}) < \Delta\Theta_1 < \Delta\theta_{1,j}\,|\,\Theta_2 = \theta_2,\, \Theta_3 = \theta_3, \ldots,\, \Theta_m = \theta_m\right\} \\
+ &\sum_{j\in\mathcal{J}^+} \Pr\left\{\Delta\theta_{1,j} < \Delta\Theta_1 < \Delta\theta_{1,j} + \Delta(\Delta\theta_{1,j})\,|\,\Theta_2 = \theta_2,\, \Theta_3 = \theta_3, \ldots,\, \Theta_m = \theta_m\right\},
\end{aligned} \tag{80}$$

where $\Delta\theta_{1,j},\ j = 1, \ldots, t$ are the roots of the equation $\pi_i = p_i(\Delta\theta_1)$, with $p_i(\Delta\theta_1)$ defined in (38), and

$$\mathcal{J}^+ = \left\{j\,:\,p_i'(\Delta\theta_{1,j}) > 0\right\},\ \mathcal{J}^- = \left\{j\,:\,p_i'(\Delta\theta_{1,j}) < 0\right\}. \tag{81}$$

It follows that $\Delta(\Delta\theta_{1,j}) > 0\ \forall j \in \mathcal{J}^+$ and similarly, $\Delta(\Delta\theta_{1,j}) < 0\ \forall j \in \mathcal{J}^-$. Using this fact and the independence of the Θ_i's, we can simplify (80) as

$$\begin{aligned}
&\Pr\left\{\pi_i \le \Pi_i \le \pi_i + \Delta(\pi_i)\,|\,\Theta_2 = \theta_2,\, \Theta_3 = \theta_3, \ldots,\, \Theta_m = \theta_m\right\} \\
&\quad= \sum_{j\in\mathcal{J}^-} \Pr\left\{\Delta\theta_{1,j} - |\Delta(\Delta\theta_{1,j})| < \Delta\Theta_1 < \Delta\theta_{1,j}\right\} \\
&\quad+ \sum_{j\in\mathcal{J}^+} \Pr\left\{\Delta\theta_{1,j} < \Delta\Theta_1 < \Delta\theta_{1,j} + |\Delta(\Delta\theta_{1,j})|\right\}.
\end{aligned} \tag{82}$$

[6]The operator $\Delta(x)$ denotes an incremental change (possibly negative) in the quantity x

Further, since $\Pr\{x \le X \le x + |\Delta(x)|\} = \Pr\{x - |\Delta(x)| \le X \le x\} \approx f_X(x) \cdot |\Delta(x)|$, it follows from (82) that

$$\begin{aligned} f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i \,|\, \theta_2, \ldots, \theta_m) \cdot \Delta\pi_i &= \sum_{j \in \mathcal{J}^-} f_{\Delta\Theta_1}(\Delta\theta_{1,j}) \cdot |\Delta(\Delta\theta_{1,j})| \\ &+ \sum_{j \in \mathcal{J}^+} f_{\Delta\Theta_1}(\Delta\theta_{1,j}) \cdot |\Delta(\Delta\theta_{1,j})| \\ &= \sum_{j=1}^{t} f_{\Delta\Theta_1}(\Delta\theta_{1,j}) \cdot |\Delta(\Delta\theta_{1,j})|. \end{aligned} \tag{83}$$

By construction, $\Delta\pi_i > 0$, which implies $|\Delta\pi_i| = \Delta\pi_i$, and therefore

$$\begin{aligned} f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i \,|\, \theta_2, \ldots, \theta_m) &= \sum_{j=1}^{t} f_{\Delta\Theta_1}(\Delta\theta_{1,j}) \cdot \frac{|\Delta(\Delta\theta_{1,j})|}{|\Delta\pi_i|} \\ &= \sum_{j=1}^{t} f_{\Delta\Theta_1}(\Delta\theta_{1,j}) \cdot \left| \frac{\Delta\pi_i}{\Delta(\Delta\theta_{1,j})} \right|^{-1}. \end{aligned} \tag{84}$$

In the limit, as $\Delta(\Delta\theta_{1,j}) \to 0$,

$$\begin{aligned} f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i \,|\, \theta_2, \ldots, \theta_m) &= \sum_{j=1}^{t} f_{\Delta\Theta_1}(\Delta\theta_{1,j}) \cdot \lim_{\Delta(\Delta\theta_{1,j}) \to 0} \left| \frac{\Delta\pi_i}{\Delta(\Delta\theta_{1,j})} \right|^{-1} \\ &= \sum_{j=1}^{t} f_{\Delta\Theta_1}(\Delta\theta_{1,j}) \cdot \left| \lim_{\Delta(\Delta\theta_{1,j}) \to 0} \frac{\Delta\pi_i}{\Delta(\Delta\theta_{1,j})} \right|^{-1} \\ &= \sum_{j=1}^{t} \frac{f_{\Delta\Theta_1}(\Delta\theta_{1,j})}{|p_i'(\Delta\theta_{1,j})|}. \end{aligned} \tag{85}$$

C. Dependent model parameters

In order to obtain $f_{\Pi_i}(\pi_i)$, (pdfs of the other indices follow similarly), we first compute $f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i|\theta_2, \ldots, \theta_m)$ as follows:

$$f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i|\theta_2, \ldots, \theta_m) = \sum_{j=1}^{r} \frac{f_{\Delta\Theta_1|\Theta_2,\ldots,\Theta_m}(\Delta\theta_{1,j}|\theta_2, \ldots, \theta_m)}{|p_i'(\Delta\theta_{1,j})|}, \tag{86}$$

where $\Delta\theta_{1,1}, \Delta\theta_{1,2}, \ldots \Delta\theta_{1,r}$ are the $r \le t$ real roots of $\pi_i = \pi_i(m_\Theta) + \sum_{k=1}^{t} \frac{b_{ki}}{k!} \Delta\Theta_1^k$, and $m_\Theta = [m_{\Theta_1}, \theta_2, \ldots, \theta_m]$. Note that the above result follows from (82). Since the model parameters are dependent, the numerator in (86) does not simplify to $f_{\Delta\Theta_1}(\Delta\theta_{1,j})$ (as was the case in (39)). If the joint pdf of the model parameters is known, $f_{\Delta\Theta_1|\Theta_2,\ldots,\Theta_m}(\Delta\theta_{1,j}|\theta_2,\ldots,\theta_m)$ can be obtained as follows

$$\begin{aligned} f_{\Delta\Theta_1|\Theta_2,\ldots,\Theta_m}(\Delta\theta_1|\theta_2,\ldots,\theta_m) &= f_{\Theta_1|\Theta_2,\ldots,\Theta_m}(m_{\Theta_1}+\Delta\theta_1|\theta_2,\ldots,\theta_m) \\ &= \frac{f_{\Theta_1,\Theta_2,\ldots,\Theta_m}(m_{\Theta_1}+\Delta\theta_1,\theta_2,\ldots,\theta_m)}{f_{\Theta_2,\ldots\Theta_m}(\theta_2,\ldots,\theta_m)} \\ &= \frac{f_{\Theta_1,\Theta_2,\ldots,\Theta_m}(m_{\Theta_1}+\Delta\theta_1,\theta_2,\ldots,\theta_m)}{\int_{\theta_1} f_{\Theta_1,\Theta_2,\ldots,\Theta_m}(\theta_1,\theta_2,\ldots,\theta_m)d\theta_1}. \end{aligned} \tag{87}$$

The last step in the derivation above is necessary, since we assume only the joint distribution is known. Once $f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i|\theta_2,\ldots,\theta_m)$ is computed, it is straightforward to obtain $f_{\Pi_i}(\pi_i)$ from the total probability theorem as follows:

$$\begin{aligned} f_{\Pi_i}(\pi_i) &= \int_{\theta_2}\cdots\int_{\theta_m} f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i|\theta_2,\ldots,\theta_m)\, f_{\Theta_2,\ldots\Theta_m}(\theta_2,\ldots\theta_m)d\theta_2\ldots d\theta_m \\ &= \int_{\theta_2}\cdots\int_{\theta_m} f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i|\theta_2,\ldots,\theta_m)\left(\int_{\theta_1} f_{\Theta_1,\ldots,\Theta_m}(\theta_1,\ldots\theta_m)d\theta_1\right)d\theta_2\ldots d\theta_m \end{aligned} \tag{88}$$

Since $f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i|\theta_2,\ldots,\theta_m)$ does not depend on θ_1, we can express (88) as follows:

$$f_{\Pi_i}(\pi_i) = \int_{\theta_1}\int_{\theta_2}\cdots\int_{\theta_m} f_{\Pi_i|\Theta_2,\ldots,\Theta_m}(\pi_i|\theta_2,\ldots,\theta_m)\, f_{\Theta_1,\ldots\Theta_m}(\theta_1,\ldots\theta_m)d\theta_1 d\theta_2\ldots d\theta_m. \tag{89}$$

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