A Stochastic Hybrid Systems Framework for Analysis of Markov Reward Models

S. V. Dhople[a,1,*], L. DeVille[c,2], A. D. Domínguez-García[b,1]

[a] *Department of Electrical and Computer Engineering,*
University of Minnesota, Minneapolis, MN 55455
[b] *Department of Electrical and Computer Engineering,*
University of Illinois at Urbana-Champaign, Urbana, IL 61801
[c] *Department of Mathematics,*
University of Illinois at Urbana-Champaign, Urbana, IL 61801

Abstract

In this paper, we propose a framework to analyze Markov reward models, which are commonly used in system performability analysis. The framework builds on a set of analytical tools developed for a class of stochastic processes referred to as Stochastic Hybrid Systems (SHS). The state space of an SHS is comprised of: i) a discrete state that describes the possible configurations/modes that a system can adopt, which includes the nominal (non-faulty) operational mode, but also those operational modes that arise due to component faults, and ii) a continuous state that describes the reward. Discrete state transitions are stochastic, and governed by transition rates that are (in general) a function of time and the value of the continuous state. The evolution of the continuous state is described by a stochastic differential equation and reward measures are defined as functions of the continuous state. Additionally, each transition is associated with a reset map that defines the mapping between the pre- and post-transition values of the discrete and continuous states; these mappings enable the definition of impulses and losses in the reward. The proposed SHS-based framework unifies the analysis of a variety of previously studied reward models. We illustrate the application of the framework to performability analysis via analytical and numerical examples.

Keywords: Markov reliability models, Reward models, Performability analysis, Stochastic hybrid systems.

1. Introduction

Continuous-time Markov chains (CTMCs) are commonly used for system reliability/availability modeling in many application domains, including: computer systems [9, 28, 41], communication networks [27, 29],

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*Corresponding author
Email addresses: `sdhople@UMN.EDU` (S. V. Dhople), `rdeville@ILLINOIS.EDU` (L. DeVille), `aledan@ILLINOIS.EDU` (A. D. Domínguez-García)

electronic circuits [5, 42], power and energy systems [1, 2], and phased-mission systems [25, 47]. A Markov reward model is defined by a CTMC, and a reward function that maps each element of the Markov chain state space into a real-valued quantity [32, 40, 45]. The appeal of Markov reward models is that they provide a unified framework to define and evaluate reliability/availability measures that capture system performance measures of interest; in the literature, this is typically termed *performability analysis* [30, 31, 38, 40, 46, 18, 26, 17]. In this paper, we propose a framework that enables the formulation of very general reward models, and unifies the analysis of a variety of previously studied Markov reward models. The framework foundations are a set of theoretical tools developed to analyze a class of stochastic processes referred to as Stochastic Hybrid Systems (SHS) [21], which are a subset of the more general class of stochastic processes known as Piecewise-Deterministic Markov processes [8].

The state space of an SHS is comprised of a *discrete state* and a *continuous state*; the pair formed by these is what we refer to as the *combined state* of the SHS. The transitions of the discrete state are stochastic, and the rates at which these transitions occur are (in general) a function of time, and the value of the continuous state. For each value that the discrete state takes, the evolution of the continuous state is described by a stochastic differential equation (SDE). The SDEs associated with each value that the discrete state takes need not be the same; indeed, in most applications they differ significantly. Additionally, each discrete-state transition is associated with a reset map that defines how the pre-transition discrete and continuous states map into the post-transition discrete and continuous states. Within the context of performability modeling, the set in which the discrete state takes values describes the possible configurations/modes that a system can adopt, which includes the nominal (non-faulty) operational mode, but also those operational modes that arise due to faults (and repairs) in the components that comprise the system. The continuous state captures the evolution of some variables associated with system performance, and as such, can be used to define reward measures that capture a particular performance measure of interest. Finally, the reset maps can define instantaneous gains and losses in reward measures that result from discrete-state transitions associated with failures/repairs.

In order to fully characterize an SHS-based reward model, we need to obtain the distribution of the combined state. However, this is an intractable problem in general, due to the coupling between the evolution of the discrete and continuous states and the presence of reset maps. In fact, this problem can only be solved in a few special cases. For instance, if we assume that the discrete state does not depend on the continuous state, the evolution of the former can be written as a CTMC; and as such, its probability distribution is fully characterized by the solution of the Chapman-Kolmogorov equations. However, unless we also assume that the resets do not change the value of the continuous state, it is not straightforward to obtain the continuous-state probability distribution. Given the difficulty in obtaining the distribution of the combined state, we settle for a method that allows the computation of any arbitrary number of their moments. To this end, we rely on the extended generator of the SHS, which together with Dynkin's formula can be

used to obtain a differential equation that describes the evolution of the expectation of any function of the combined state, as long as such a function is in the domain of the extended generator. Following the approach outlined in [20, 21], we show that under certain general assumptions, monomial functions are always in the domain of the extended generator, and thus, Dynkin's formula holds. Additionally, for SHS where the reset maps, transition rates, and the vector fields defining the SDEs are polynomial, the generator maps the set of monomial functions to itself. Therefore, Dynkin's formula gives a closed set of ordinary differential equations (ODEs) that describes the evolution of each moment in terms of the values of the other moments. Since there are infinitely many monomial functions, this formally produces an infinite-dimensional system of ODEs in what is referred to in the stochastic process literature as a *closure problem*.

The examples and case studies in this work demonstrate how the proposed SHS-based framework applies to reward models where the rate at which the reward grows is: i) constant—this case is referred as the *rate reward model* [39], ii) governed by a first-order linear differential equation—we refer to this case as a *first-order reward model*, and iii) governed by a linear SDE—this case is referred as the *second-order reward model* [3, 23]. As demonstrated in Section 3.1, the SHS-based framework can specify even more general reward models, but we restrict our attention to the above cases as they have been previously studied in the literature; this allows us to validate and verify our results. We will show that the structure of the standard reward models described above is such that there are finite-dimensional truncations of the ODEs governing the moment evolution that are closed, i.e., there are finite subsets of moments such that the evolution of any member of this subset is a function only of the other members of this subset. In other words, these conventional reward models do not lead to a closure problem, and we only have to solve a finite-dimensional ODE to determine the evolution of the reward moments.

Several numerical methods have been proposed to compute the reward distributions for rate reward models (see, e.g., [15, 36, 38, 33, 43, 6] and the references therein). However, for more general reward models, e.g., second-order reward models with impulses and/or losses in the accumulated reward, it is very difficult to obtain explicit, closed-form, analytical solutions for the partial differential equations (PDEs) that describe the evolution of the reward distributions [22]. In practice, in order to analyze such reward models, numerical methods are utilized to integrate the PDEs governing the evolution of the accumulated reward probability density function [10, 22] (see also [11, 44] for discussions on specific reward modeling and analysis software packages). It is worth noting that systems with deterministic flows and random jumps in the state have been widely studied in the nuclear engineering community (in light of the description above, these are a type of SHS). For instance, Chapman-Kolmogorov equations with appropriate Markovian assumptions are utilized to derive the PDEs that govern the continuous states in [13, 12, 14]. However, even in this body of work, it has been acknowledged that closed-form analytical solutions to the PDEs can be derived only for simple models [12].

An alternative to numerical integration for characterizing the distribution of the reward is to compute its

moments, which then can be used, e.g., to compute bounds on the probabilities of different events of interest using probability inequalities. In this regard, a number of methods have been proposed in the literature for computing moments in reward models. For example, techniques based on the Laplace transform of the accumulated-reward distribution are proposed in [16, 23, 24, 39]. In [32], the first moment of the accumulated reward in these models is computed following a method based on the frequency of transitions in the underlying Markov chain. A numerical procedure based on the uniformization method is proposed to compute the moments of the accumulated reward in [7]. Methods from calculus of variations are used to derive differential equations that provide moments of rewards for rate-reward models in [37]. In the same vein of these earlier works, the SHS-based framework proposed in this paper provides a method to compute any desired number of reward moments. The advantages of the SHS approach are twofold: i) it provides a unified framework to describe and analyze a wide variety of reward models (even beyond the rate-, first-, and second-order reward models that our case studies focus on), and ii) the method is computationally efficient as it involves solving a linear ODE, for which there are very efficient numerical integration methods.

The remainder of this paper is organized as follows. In Section 2, we provide a brief overview of Markov reliability and reward models. In Section 3, we describe fundamental notions of SHS, and demonstrate how the Markov reward models studied in this work are a type of SHS. Case studies are discussed in Section 4, while Section 5 illustrates the moment closure problem in SHS. Concluding remarks and directions for future work are described in Section 6.

2. Preliminaries

In this section, we provide a brief overview of Markov reliability and reward models, while in the process, we introduce some relevant notation and terminology used throughout the paper. For a detailed account on these topics, interested readers are referred to [40].

2.1. Markov Reliability Models

Let $Q(t)$ denote a stochastic process taking values in a finite set $\mathcal{Q}$; the elements in this set index the system *operational modes*, including the nominal (non-faulty) mode and the modes that arise due to faults (and repairs) in the components comprising the system. The stochastic process $Q(t)$ is called a Continuous-Time Markov Chain (CTMC) if it satisfies the Markov property, which is to say that

$$\Pr\left\{Q(t_r) = i | Q(t_{r-1}) = j_{r-1}, \ldots, Q(t_1) = j_1\right\} = \Pr\left\{Q(t_r) = i | Q(t_{r-1}) = j_{r-1}\right\}, \tag{1}$$

for $t_1 < \cdots < t_r$, $\forall i, j_1, \ldots, j_{r-1} \in \mathcal{Q}$ [19]. The chain Q is said to be *homogeneous* if it satisfies

$$\Pr\left\{Q(t) = i | Q(s) = j\right\} = \Pr\left\{Q(t - s) = i | Q(0) = j\right\}, \ \forall i, j \in \mathcal{Q}, 0 < s < t. \tag{2}$$

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Homogeneity of $Q(t)$ implies that the times between transitions (i.e., the sojourn times in the states) are exponentially distributed.

Denote the probability that the chain is in state i at time $t \geq 0$ by $\pi_i(t) := \Pr\{Q(t) = i\}$, and the entries of the row vector of occupational probabilities by $\{\pi_q(t)\}_{q \in \mathcal{Q}}$. The evolution of $\pi(t)$ is governed by the Chapman-Kolmogorov equations:

$$\dot{\pi}(t) = \pi(t)\Lambda, \tag{3}$$

where $\Lambda \in \mathbb{R}^{|\mathcal{Q}| \times |\mathcal{Q}|}$ is the Markov chain generator matrix whose entries are obtained from the failure and repair rates of the system components. In the context of this work, the CTMC defined in (3) describes a Markov reliability model.

2.2. Markov Reward Models

A Markov reward model is comprised of a Markov chain $Q(t)$ taking values in the set $\mathcal{Q}$ (which, as stated previously, describes the possible system operational modes,) and an *accumulated reward* $Y(t)$, which captures some performance measure of interest. The most commonly studied Markov reward models are rate-reward models and second-order reward models (see, e.g., [22, 23], and the references therein). The accumulated reward in rate-reward models evolves according to

$$\begin{aligned} \frac{dX(t)}{dt} &= a(Q(t)), \\ Y(t) &= X(t), \end{aligned} \tag{4}$$

where $a \colon \mathcal{Q} \to \mathbb{R}$ is the (discrete-state-dependent) reward growth rate. In second-order reward models, the accumulated reward evolves according to

$$\begin{aligned} dX(t) &= a(Q(t))\,dt + c(Q(t))\,dW_t, \\ Y(t) &= X(t), \end{aligned} \tag{5}$$

where $a \colon \mathcal{Q} \to \mathbb{R}$, $c \colon \mathcal{Q} \to \mathbb{R}$, and $W_t \colon \mathbb{R}^+ \to \mathbb{R}$ is the Wiener process. Impulses in the accumulated reward capture one-time effects due to failures/repairs of components in the system. As described in Section 1, various methods have been proposed to tackle impulses in rate reward models.

3. Stochastic Hybrid Systems Formalism for Markov Reward Models Analysis

This section begins with a brief overview on the most general class of reward models that can be described using the SHS formalism. Then, through straightforward simplifications, we recover Markov reward models previously proposed in the literature. The main result presented in this section, which is adopted from [20, 21], establishes that, for all the aforementioned reward models, it is possible to describe the evolution of the accumulated-reward conditional moments by a set of ODEs. Then, the accumulated reward moments can be obtained by applying the law of total expectation.

3.1. General Reward Models Defined as SHS

In the most general sense, an SHS is a combination of a continuous-time, discrete-state stochastic process $Q(t) \in \mathcal{Q}$, coupled with a continuous-time, continuous-state stochastic process $X(t) \in \mathbb{R}^d$. Additionally, we assume that this system is fully coupled, in the sense that the evolution of the continuous state depends on the current value of the discrete state, and the transitions of the discrete state depend on the current value of the continuous state. In the context of this paper, in addition to the two processes described above, we include a third process $Y(t)$, $Y : \mathbb{R}^+ \to \mathbb{R}$ obtained as a function of $X(t)$; this third process enables the definition of different reward measures.

We now give an intuitive, non-rigorous description of a general SHS; see AppendixA for a mathematically rigorous definition. First, we define the functions

$$\lambda_j : \mathcal{Q} \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^+, \quad \phi_j : \mathcal{Q} \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathcal{Q} \times \mathbb{R}^d, \quad j \in \mathcal{J}, \tag{6}$$

which we call the *transition rates* and the *transition reset maps*, respectively. The idea of these functions is that at any time t, if the system is in state $(Q(t), X(t))$, it undergoes transition j with rate $\lambda_j(Q(t), X(t), t)$, and if it undergoes this transition, then it instantaneously applies the map $\phi_j(Q(t), X(t), t)$ to the current values of $Q(t)$ and $X(t)$, and discontinuously changes their values at that moment.

More specifically, for any time $t > 0$, we say that the probability of transition j occurring in the time domain $[t, t + \Delta t)$ is

$$\lambda_j(Q(t), X(t), t)\Delta t + o(\Delta t), \tag{7}$$

and if it does occur, then we define

$$(Q(t + \Delta t), X(t + \Delta t)) = \phi_j(Q((t + \Delta t)^-), X((t + \Delta t)^-), t + \Delta t), \tag{8}$$

thus obtaining new values for $Q(t)$ and $X(t)$.[3] From this, we see that the probability of *no* transition occurring in $[t, t + \Delta t)$ is $1 - \Delta t \sum_{j \in \mathcal{J}} \lambda_j(Q(t), X(t), t)$. Finally, between transitions, we prescribe that $X(t), Y(t)$ evolve according to

$$\begin{aligned} dX(t) &= f(Q(t), X(t), t) \, dt + g(Q(t), X(t), t) \, dW_t, \\ Y(t) &= h(Q(t), X(t), t), \end{aligned} \tag{9}$$

where $W_t : \mathbb{R}^+ \to \mathbb{R}^l$ is the l-dimensional Wiener process, $f : \mathcal{Q} \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$, $g : \mathcal{Q} \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^{d \times l}$, and $h : \mathcal{Q} \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$.

The main reason that the above description is only intuitive is that it is only an asymptotic description, and it is not *a priori* clear at this stage that a sensible limit exists when Δt is taken to zero. In fact, such

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[3]We use the notation $a(t^-) = \lim_{s \nearrow t} a(s)$ to denote the left-hand limit of the function a.

a limit does exist (as we show in AppendixA), and, moreover, the asymptotic description given above is a standard method one would use to numerically simulate realizations of the stochastic process (as is done in the case studies in Section 4).

For the class of SHS studied in [20], the vector fields that govern the evolution of the continuous state (f, g, and h), the reset maps (ϕ_j), and the transition rates (λ_j), are required to be polynomial functions of the continuous state. In general, as illustrated in Section 5, the evolution of the moments of the continuous state is governed by an infinite-dimensional system of ODEs, and moment closure methods have to be applied to obtain truncated state-space descriptions [21]. For the numerical examples of the reward models we study in this paper, the vector fields that govern the evolution of the continuous state and the reset maps are linear, and, moreover, the transition rates are not assumed to be functions of the continuous state. As we show below, this implies that the differential equations that govern the evolution of the conditional moments in these models are effectively finite dimensional, and moment-closure methods are unnecessary.

3.2. Markov Reward Models Defined as SHS

Although the formalism outlined in Section 3.1 provides a unified and generalized modeling framework to tackle a wide variety of reward models, in the remainder of the paper we restrict our attention to a class of Markov reward models that can be formulated as a special case of this general SHS model. In particular, we assume that i) the SDEs describing the evolution of the continuous state are linear (or, more precisely, affine) in the continuous state $X(t)$, ii) the transition rates governing the jumps of $Q(t)$ are independent of $X(t)$, and iii) the reward $Y(t)$, is a linear function of the continuous state $X(t)$. More precisely, the SDE governing $X(t)$ (and therefore $Y(t)$) is given by

$$
\begin{aligned}
dX(t) &= A(Q(t),\,t)\,X(t)dt + B(Q(t),\,t)dt + C(Q(t),\,t)\,dW_t, \\
Y(t) &= R(Q(t),\,t)\,X(t),
\end{aligned}
\tag{10}
$$

where $W_t : \mathbb{R}^+ \to \mathbb{R}^l$ is the l-dimensional Wiener process, $A : \mathcal{Q} \times \mathbb{R}^+ \to \mathbb{R}^{d \times d}$, $B : \mathcal{Q} \times \mathbb{R}^+ \to \mathbb{R}^d$, $C : \mathcal{Q} \times \mathbb{R}^+ \to \mathbb{R}^{d \times l}$, and $R : \mathcal{Q} \times \mathbb{R}^+ \to \mathbb{R}^{1 \times d}$.

We first note that under these assumptions, the discrete process $Q(t)$ is a CTMC—in particular, one can understand the pathwise evolution of $Q(t)$ without knowing $X(t), Y(t)$. If we further assume that the transition rates are not a function of time, i.e., if $\lambda_j \colon \mathcal{Q} \to \mathbb{R}^+$, the Markov chain is homogeneous. In the context of this work, and as discussed in Section 2, the CTMC $Q(t)$ describes a Markov reliability model, while $(Q(t),\, X(t),\, Y(t))$ describes a Markov reward model.

It should be noted that rate, first-order, and second-order reward models are all subsumed in this framework. In fact, to realize rate reward models, we choose $A = C = 0$ in (10); to realize first-order reward models, we choose $C = 0$ in (10). Expressed as such, (10) describes a second-order reward model; this is the most general model we explore in the numerical examples of Section 4.
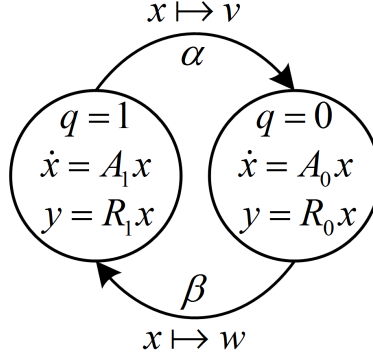
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Figure 1: State-transition diagram for the Markov reward model studied in Example 1.

The results presented in [21] for SHS apply directly to the Markov reward models examined in this work. Of particular interest is the method to obtain the moments of $X(t)$ (from which we can recover the accumulated-reward moments). As described subsequently in Section 3.3, this method is based on defining appropriate test functions and formulating the extended generator for the underlying stochastic processes. We end this section by illustrating the notation introduced so far with a simple example. We will revert to this example in Section 3.4 to demonstrate how the moments of the accumulated reward are obtained from appropriately defined test functions.

Example 1. *Consider the Markov reward model in Fig. 1, which is commonly obtained by aggregation of many-state CTMCs [9]. In this model, the underlying CTMC $Q(t)$ takes value 0 whenever the system is in a failed mode, and takes values 1 whenever the system is operational. Associated with this Markov chain, we consider a first-order reward model. To this end, define $X(t) = [X_1(t), X_2(t), \ldots, X_d(t)]^T$, which evolves according to*

$$\frac{dX(t)}{dt} = A(Q(t))X(t) =: A_{Q(t)}X(t), \tag{11}$$

where $A_{Q(t)} = A_0 \in \mathbb{R}^{d \times d}$ if $Q(t) = 0$, and $A_{Q(t)} = A_1 \in \mathbb{R}^{d \times d}$ if $Q(t) = 1$. The accumulated reward $Y(t)$ is given by $Y(t) = R(Q(t))X(t) =: R_{Q(t)}X(t)$, where $R_{Q(t)} = R_0 \in \mathbb{R}^{1 \times d}$ if $Q(t) = 0$, and $R_{Q(t)} = R_1 \in \mathbb{R}^{1 \times d}$ if $Q(t) = 1$. Now choose two numbers $\alpha, \beta \in \mathbb{R}^+$ and two vectors $v, w \in \mathbb{R}^d$. Basically, α, v will govern the transitions from state $1 \to 0$, so that we transition from operational mode 1 to 0 with (failure) rate α, and when we do so, we reset the value of $X(t)$ to v, and similarly for β, w in the other direction. Following the notation introduced in Section 3.1, define the set of transitions by $\mathcal{J} = \{0, 1\}$, with transition rates[4]

$$\lambda_0(q, x) = \delta_{q,1}\alpha, \quad \lambda_1(q, x) = \delta_{q,0}\beta, \tag{12}$$

and reset maps

$$\phi_0(q, x) = (0, \delta_{q,1}v), \quad \phi_1(q, x) = (1, \delta_{q,0}w). \tag{13}$$

It turns out that there is a more compact way to formulate this in the SHS-based framework. To this end, we can say that there is exactly one transition, and define the following transition rate and reset map

$$\lambda(q, x) = \begin{cases} \alpha, & q = 1, \\ \beta, & q = 0. \end{cases}, \quad \phi(q, x) = \begin{cases} (0, v), & q = 1, \\ (1, w), & q = 0, \end{cases} \tag{14}$$

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[4]In subsequent developments we use standard Kronecker delta notation, i.e., $\delta_{i,j} = 1$ if $i = j$ and $= 0$ if $i \neq j$.

which can be written more compactly using the Kronecker delta notation as follows:

$$\lambda(q, x) = \delta_{q,1}\alpha + \delta_{q,0}\beta, \quad \phi(q, x) = (1 - q, \delta_{q,1}v + \delta_{q,0}w). \tag{15}$$

These models are equivalent with probability one, since a transition with a zero rate occurs with probability zero (see AppendixA for details). Also, note that as long as we assume that our model has only one way to transition out of any given discrete state, then we can always represent our system with one transition. □

3.3. Test Function and Extended Generator of the Stochastic Processes

For the reward model introduced in (10), define a *test function* $\psi(q, x, t)$, $\psi : \mathcal{Q} \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$, where q represents the discrete state of the CTMC, and x represents the continuous state from which the accumulated reward is recovered. The *extended generator* (referred interchangeably as *generator* subsequently) is denoted by $(L\psi)(q, x, t)$, and defined as

$$
\begin{aligned}
(L\psi)(q, x, t) \quad := \quad & \frac{\partial}{\partial x}\psi(q, x, t) \cdot (A(q, t)x + B(q, t)) + \frac{\partial}{\partial t}\psi(q, x, t) \\
+ \quad & \frac{1}{2}\sum_{i,j}\left((CC^T)_{i,j}(q,t)\frac{\partial^2}{\partial x_i x_j}\psi(q, x, t)\right) \\
+ \quad & \sum_{j \in \mathcal{J}}\lambda_j(q, x, t)\left(\psi\left(\phi_j\left(q, x, t\right)\right) - \psi(q, x, t)\right),
\end{aligned} \tag{16}
$$

where $\partial\psi/\partial x \in \mathbb{R}^{1 \times d}$ and $\partial^2\psi/\partial x^2 \in \mathbb{R}^{d \times d}$ denote the gradient and Hessian of $\psi(q, x, t)$ with respect to x, respectively; and the summation (in the third line) is over all transitions of the underlying CTMC [8, 21]. The evolution of the expected value of the test function $\mathbb{E}\left[\psi(Q(t), X(t), t)\right]$, is governed by Dynkin's formula, which can be stated in differential form [8, 21] as follows:

$$\frac{d}{dt}\mathbb{E}[\psi(Q(t), X(t), t)] = \mathbb{E}[(L\psi)(Q(t), X(t), t)]. \tag{17}$$

Said in words, (17) implies that the time rate of change of the expected value of a test function evaluated on the stochastic process is given by the expected value of the generator. We also point out that the formula for the generator is plausible on intuitive grounds: if we "turn off" the transitions between discrete states and only had a simple diffusion, then the generator would be the first two lines. In contrast, if we "turn off" the SDE evolution for the continuous state, then the transitions between modes is generated by the third line. An appeal to the infinitesimal nature of (17) and linearity tells us that we should add these two operators together to obtain the generator for the entire process. This argument is informal and intuitive; a rigorous argument demonstrating (17) for SHS can be found (in various forms) in [8, 21, 35].

Dynkin's formula holds for every ψ that is in the domain of the extended generator L. We point out that in the current work, we will only consider those ψ that do not explicitly depend on time, and so the second term in line one of (16) does not appear. Describing the domain of this operator is, in general, technically difficult [8]. However, following [21], we show in AppendixB that in the current framework (namely, SDEs that have affine drifts with additive noise, and state-independent transition rates), all functions polynomial

in x are in the domain of L and, moreover, that Dynkin's formula holds for all such polynomials.

3.4. Recovering Differential Equations for Conditional Moments from Test Functions

Next, we summarize the procedure outlined in [21] to specify a family of test functions from which the moments of the accumulated reward can be recovered by using (16) and (17). For a Markov reward model where the underlying CTMC takes values in the set $\mathcal{Q}$, we define a family of test functions of the form

$$\psi_i^m(q, \, x) := \delta_{i,q} x^m = \begin{cases} x^m & \text{if} \quad q = i \\ 0 & \text{if} \quad q \neq i \end{cases}, \, \forall i \in \mathcal{Q}, \tag{18}$$

where $m := (m_1, \, m_2, \ldots, m_d) \in \mathbb{N}^{1 \times d}$, and $x^m := x_1^{m_1} x_2^{m_2} \ldots x_d^{m_d}$. We also define the *conditional moments* at time t, $\mu_i^{(m)}(t)$, $\forall i \in \mathcal{Q}$, by

$$\mu_i^{(m)}(t) := \mathbb{E}\left[\psi_i^{(m)}(q, \, x)\right] = \mathbb{E}\left[X^m(t) | Q(t) = i\right] \cdot \Pr\left\{Q(t) = i\right\}, \tag{19}$$

and for every $m \in \mathbb{N}^{1 \times d}$, the entries of the (row) *vector of conditional moments* are denoted by $\{\mu_q^{(m)}(t)\}_{q \in \mathcal{Q}}$. The last equality in (19) follows from the definition of the test functions in (18). By appropriately picking the m_i's, we can isolate the conditional moments of interest. We demonstrate this next, in the context of the system considered in Example 1.

Example 2. *Recall the Markov reward model introduced in Example 1; associated with the two discrete states, define the following test functions*

$$\begin{aligned} \psi_0^{(m)}(q, \, x) &= \delta_{q,0} x^m = \begin{cases} x^m & \textit{if} \quad q = 0 \\ 0 & \textit{if} \quad q = 1 \end{cases}, \\ \psi_1^{(m)}(q, \, x) &= \delta_{q,1} x^m \begin{cases} 0 & \textit{if} \quad q = 0 \\ x^m & \textit{if} \quad q = 1 \end{cases}, \end{aligned} \tag{20}$$

where $m \in \mathbb{N}^{1 \times d}$ and $x^m = x_1^{m_1} x_2^{m_2} \ldots x_d^{m_d}$. As stated previously, by appropriately picking m, we can recover many conditional moments of interest. For instance, note that choosing $m = (0, 0, \ldots, 0)$ recovers the discrete-state occupational probabilities

$$\mu_i^{(0,0,\ldots,0)}(t) = \Pr\left\{Q(t) = i\right\} = \pi_i(t). \tag{21}$$

Similarly, picking $m = (2, 0, \ldots, 0)$ isolates the second-order conditional moment of $X_1(t)$:

$$\begin{aligned} \mu_i^{(2,0,\ldots,0)}(t) &= \mathbb{E}\left[X^{(2,0,\ldots,0)}(t) | Q(t) = i\right] \cdot \Pr\left\{Q(t) = i\right\} \\ &= \mathbb{E}\left[X_1^2(t) | Q(t) = i\right] \cdot \pi_i(t). \end{aligned} \tag{22}$$

Finally, picking $m = (1, 1, \ldots, 1)$ yields the conditional expectation of the product $\prod_{\ell=1}^d X_\ell(t)$:

$$\begin{aligned} \mu_i^{(1,1,\ldots,1)}(t) &= \mathbb{E}\left[X^{(1,1,\ldots,1)}(t) | Q(t) = i\right] \cdot \Pr\left\{Q(t) = i\right\} \\ &= \mathbb{E}\left[\prod_{\ell=1}^d X_\ell(t) | Q(t) = i\right] \cdot \pi_i(t). \end{aligned} \tag{23}$$

Other moments of interest can be recovered similarly. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.5. Evolution of the Accumulated Reward

For a given m (which, as shown previously, can be defined to isolate the conditional moment of interest), we apply (16) to obtain expressions for the extended generators, $\left(L\psi_i^{(m)}\right)(q,\,x)$, $i \in \mathcal{Q}$. From Dynkin's formula in (17), we then obtain a set of differential equations that govern the conditional moments:

$$\frac{d}{dt}\mu_i^{(m)}(t) = \frac{d}{dt}\mathbb{E}\left[\psi_i^{(m)}(q,\,x)\right] = \mathbb{E}\left[\left(L\psi_i^{(m)}\right)(q,\,x)\right],\ \forall i \in \mathcal{Q}. \tag{24}$$

The problem of interest is to obtain the p-order moment of the accumulated reward $\mathbb{E}[Y^p(t)]$, from the conditional moments defined above. Recall that the accumulated reward is given by $Y(t) = R(Q(t),t)\,X(t) = \sum_{s=1}^{d} r_s(Q(t),\,t)\,X_s(t)$, which implies that $Y^p(t)$ is a polynomial function of $X_s(t)$, $s = 1,\,2,\ldots,\,d$. In particular, applying the multinomial theorem, we obtain

$$Y^p(t) = \sum_{m_1+m_2+\cdots+m_d=p} \binom{p}{m_1,\,m_2,\ldots,\,m_d} \prod_{1\le s\le d} \left(r_s(Q(t),\,t)\,X_s(t)\right)^{m_s}, \tag{25}$$

i.e., $Y^p(t)$ can be expressed as a polynomial function of $X_s(t)$, $s = 1,\,2,\ldots,\,d$. There is a more compact way to write the multinomial theorem that we will find useful subsequently. Given $m = (m_1,\ldots,m_d) \in \mathbb{N}^d$, we define

$$|m| := \sum_{s=1}^{d} m_s, \quad \binom{p}{m} := \binom{p}{m_1,\,m_2,\ldots,\,m_d}; \tag{26}$$

then (25) can be compactly expressed as

$$Y^p(t) = \sum_{|m|=p} \binom{p}{m} (R(Q(t),t)\,X(t))^m, \tag{27}$$

where we use the following notation

$$(R(Q(t),t)X(t))^m = R^m(Q(t),t)X^m(t) = (r_1(Q(t),t)X_1(t))^{m_1} \cdot \cdots \cdot (r_d(Q(t),t)X_d(t))^{m_d}$$
$$= \prod_{1\le s\le d} (r_s(Q(t),\,t)\,X_s(t))^{m_s}. \tag{28}$$

Thus, the pth order moment of Y is given by

$$
\begin{aligned}
\mathbb{E}\left[Y^p(t)\right] &= \sum_{|m|=p} \binom{p}{m} \mathbb{E}\left[(R(Q(t),t)\,X(t))^m\right] \\
&= \sum_{|m|=p} \binom{p}{m} \sum_{i \in \mathcal{Q}} \mathbb{E}\left[(R(Q(t),t)\,X(t))^m \,|Q(t)=i\right] \Pr\{Q(t)=i\} \\
&= \sum_{|m|=p} \binom{p}{m} \sum_{i \in \mathcal{Q}} (R(i,t))^m \mathbb{E}\left[X^m(t)|Q(t)=i\right] \Pr\{Q(t)=i\} \\
&= \sum_{|m|=p} \binom{p}{m} \sum_{i \in \mathcal{Q}} (R(i,t))^m \mu_i^{(m)}(t) = \sum_{i \in \mathcal{Q}} \sum_{|m|=p} \binom{p}{m} (R(i,t))^m \mu_i^{(m)}(t).
\end{aligned}
\tag{29}
$$

Therefore, to compute $\mathbb{E}\left[Y^p(t)\right]$, all we need to know are the moments $\mu_i^{(m)}(t)$ with $i \in \mathcal{Q}$ and $|m| = p$.

Remark 1. As a special case, consider the Markov reward model described by the following scalar system (i.e., $d = 1$)

$$
\begin{aligned}
dX(t) &= (a(Q(t),\,t)\,X(t)dt + b(Q(t),\,t))\,dt + C(Q(t),\,t)\,dW_t, \\
Y(t) &= r \cdot X(t),
\end{aligned}
\tag{30}
$$

where W_t is the l-dimensional Wiener process, $a: \mathcal{Q} \times \mathbb{R}^+ \to \mathbb{R}$, $b: \mathcal{Q} \times \mathbb{R}^+ \to \mathbb{R}$, $C: \mathcal{Q} \times \mathbb{R}^+ \to \mathbb{R}^{1 \times l}$, and $r \in \mathbb{R}$. Using (29), we have that

$$
\mathbb{E}\left[Y^p(t)\right] = r^p \sum_{i \in \mathcal{Q}} \mu_i^{(p)}(t),
\tag{31}
$$

which is notably simpler than (29). ■

We revert to Example 1 to illustrate how (29) applies in practice.

Example 3. *Let us consider Example 1 for the case when $d = 2$; then, the accumulated reward is given by $Y(t) = R(Q(t))X(t) = r_1(Q(t))X_1(t) + r_2(Q(t))X_2(t)$. Suppose we are interested in computing the second-order moment of the reward, $\mathbb{E}\left[Y^2(t)\right]$. Using (29), we have*

$$
\mathbb{E}\left[Y^2(t)\right] = \sum_{i=0}^{1} \left(r_1^2(i)\mu_i^{(2,\,0)}(t) + r_2^2(i)\mu_i^{(0,\,2)}(t) + 2r_1(i)r_2(i)\mu_i^{(1,\,1)}(t) \right).
\tag{32}
$$

Note that there is no technical restriction to considering higher dimensional continuous state spaces (i.e., $d > 2$), but this would give many more terms in (32). All that remains is to compute the evolution of $\mu_i^{(m)}(t)$ with $|m| = 2$, for which we use (24); this derivation is detailed next.

First, by substituting the transition rate and reset map from (15) in the definition of L from (16), we obtain two terms[5] in the generator, namely:

$$
(L\psi)(q,x) = \frac{\partial}{\partial x}\psi(q,x) \cdot A(q)x + \lambda(q,x)(\psi(\phi(q,x)) - \psi(q,x)).
\tag{33}
$$

To compute $(L\psi_i^{(m)})(q,x)$ for $|m| = 2$; we consider each term in (33) in turn. Let us first write the coordinates of $A(q)$ as

$$
A(q) = \begin{bmatrix} a_q^{11} & a_q^{12} \\ a_q^{21} & a_q^{22} \end{bmatrix},
\tag{34}
$$

---

[5]Recall here that $B = C = 0$ and ψ does not explicitly depend on time.

then, we have that

$$\frac{\partial}{\partial x}\psi_i^{(m)}(q,x) = \delta_{i,q}\begin{bmatrix} m_1 x_1^{-1} x^m \\ m_2 x_2^{-1} x^m \end{bmatrix}^T, \quad A(q)x = \begin{bmatrix} a_q^{11}x_1 + a_q^{12}x_2 \\ a_q^{21}x_1 + a_q^{22}x_2 \end{bmatrix}. \tag{35}$$

So, the first term in (33) is

$$\begin{aligned}
\frac{\partial}{\partial x}&\psi_i^{(m)}(q,x) \cdot A(q)x \\
&= \delta_{i,q}(m_1 a_q^{11} x^m + m_1 a_q^{12} x^m \frac{x_2}{x_1} + m_2 a_q^{21} x^m \frac{x_1}{x_2} + m_2 a_q^{22} x^m) \\
&= \delta_{i,q}\left((m_1 a_q^{11} + m_2 a_q^{22})x^m + m_1 a_q^{12} x^{(m_1-1,m_2+1)} + m_2 a_q^{21} x^{(m_1+1,m_2-1)}\right) \\
&= (m_1 a_i^{11} + m_2 a_i^{22})\psi_i^{(m)}(q,x) + m_1 a_i^{12}\psi_i^{(m_1-1,m_2+1)}(q,x) + m_2 a_i^{21}\psi_i^{(m_1+1,m_2-1)}(q,x). \tag{36}
\end{aligned}$$

This calculation shows us some patterns: i) the dynamics coming from the ODE between jumps does not cross-couple the discrete states (i.e., all the subscripts in this equation are the same), ii) it is the off-diagonal terms in the matrix that cross-couple the conditional moments (i.e., if A_q was diagonal, then all the superscripts in this equation would be the same), and iii) while the subtractions in the exponents might make us worry about negative-powered moments, notice that every time we subtract a power, we multiply by an m-dependent factor (e.g., if $m_1 = 0$ then the second term in the last equation is multiplied by zero even though it formally has a -1 exponent in the formula).

We now consider the second term of (33):

$$\begin{aligned}
\lambda(q,x)&(\psi_i^{(m)}(\phi(q,x)) - \psi_i^{(m)}(q,x)) \\
&= (\delta_{q,1}\alpha + \delta_{q,0}\beta)\left(\psi_i^{(m)}(1-q, \delta_{q,1}v + \delta_{q,0}w) - \psi_i^{(m)}(q,x)\right) \\
&= (\delta_{q,1}\alpha + \delta_{q,0}\beta)\left(\delta_{1-q,i}(\delta_{q,1}v^m + \delta_{q,0}w^m) - \delta_{q,i}x^m\right) \\
&= \delta_{i,0}\left(\delta_{q,1}\alpha v^m \mathbf{1}(x) - \delta_{q,0}\beta x^m\right) + \delta_{i,1}\left(\delta_{q,0}\beta w^m \mathbf{1}(x) - \delta_{q,1}\beta x^m\right) \\
&= \delta_{i,0}\left(\alpha v^m \psi_1^{(0,0)}(q,x) - \beta\psi_0^{(m)}(q,x)\right) + \delta_{i,1}\left(\beta w^m \psi_0^{(0,0)}(q,x) - \alpha\psi_1^{(m)}(q,x)\right), \tag{37}
\end{aligned}$$

where we add the $\mathbf{1}(x)$ to stress the places where the function is constant in x. The first equality in the above derivation follows from the definition of the transition rate and reset map in (15). The second equality follows from the definition of the test functions in (20). Finally, the third equality can be derived by enumerating the terms that multiply $\delta_{i,0}$ and $\delta_{i,1}$. Note that (37) works for general d and any vector m. Writing out the two cases, $i = 0, 1$, we have

$$\lambda(q,x)(\psi_0^{(m)}(\phi(q,x)) - \psi_0^{(m)}(q,x)) = \alpha v^m \psi_1^{(0,0)}(q,x) - \beta\psi_0^{(m)}(q,x), \tag{38}$$

$$\lambda(q,x)(\psi_1^{(m)}(\phi(q,x)) - \psi_1^{(m)}(q,x)) = \beta w^m \psi_0^{(0,0)}(q,x) - \alpha\psi_1^{(m)}(q,x). \tag{39}$$

Notice the effect of switching in the discrete state is that each test function is coupled to itself (negatively), and it is coupled to the constant function of the other discrete state (positively). This makes sense, since, e.g., all entrances to operational mode 0 take place deterministically with state v, and therefore this should contribute v^m to the mth moment.

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Combining (17), (36), and (37), we obtain

$$
\begin{aligned}
\frac{d}{dt}\mu_i^{(m)} &= \mathbb{E}[(L\psi_i^{(m)})(q,x)] \\
&= \mathbb{E}[(m_1 a_i^{11} + m_2 a_i^{22})\psi_i^{(m)}(q,x) + m_1 a_i^{12}\psi_i^{(m_1-1,m_2+1)}(q,x) + m_2 a_i^{21}\psi_i^{(m_1+1,m_2-1)}(q,x)] \\
&\quad + \mathbb{E}[\delta_{i,0}\left(\alpha v^m \psi_1^{(0,0)}(q,x) - \beta\psi_0^{(m)}(q,x)\right) + \delta_{i,1}\left(\beta w^m \psi_0^{(0,0)}(q,x) - \alpha\psi_1^{(m)}(q,x)\right)] \\
&= (m_1 a_i^{11} + m_2 a_i^{22})\mu_i^{(m)}(t) + m_1 a_i^{12}\mu_i^{(m_1-1,m_2+1)}(t) + m_2 a_i^{21}\mu_i^{(m_1+1,m_2-1)}(t) \\
&\quad + \delta_{i,0}\left(\alpha v^m \mu_1^{(0,0)}(t) - \beta\mu_0^{(m)}(t)\right) + \delta_{i,1}\left(\beta w^m \mu_0^{(0,0)}(t) - \alpha\mu_1^{(m)}(t)\right) \\
&= (m_1 a_i^{11} + m_2 a_i^{22})\mu_i^{(m)}(t) + m_1 a_i^{12}\mu_i^{(m_1-1,m_2+1)}(t) + m_2 a_i^{21}\mu_i^{(m_1+1,m_2-1)}(t) \\
&\quad + \delta_{i,0}\left(\alpha v^m \pi_1(t) - \beta\mu_0^{(m)}(t)\right) + \delta_{i,1}\left(\beta w^m \pi_0(t) - \alpha\mu_1^{(m)}(t)\right). \quad (40)
\end{aligned}
$$

For example, let us make the specific choice of $A_0 = \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$, $R_0 = [1, 2]$, *and* $R_1 = [-1, 5]$. *Using (40), we obtain the following set of differential equations*

$$
\begin{cases}
\dot{\mu}_0^{(2,0)}(t) = -2\mu_0^{(2,0)}(t) - 4\mu_0^{(1,1)}(t) - \beta\mu_0^{(2,0)}(t) + \alpha v_1^2 \pi_1(t) \\
\dot{\mu}_1^{(2,0)}(t) = 2\mu_1^{(2,0)}(t) - \alpha\mu_1^{(2,0)}(t) + \beta w_1^2 \pi_0(t)
\end{cases}
$$

$$
\begin{cases}
\dot{\mu}_0^{(0,2)}(t) = -6\mu_0^{(0,2)}(t) + 2\mu_0^{(1,1)}(t) - \beta\mu_0^{(0,2)}(t) + \alpha v_2^2 \pi_1(t) \\
\dot{\mu}_1^{(0,2)}(t) = -8\mu_1^{(0,2)}(t) - \alpha\mu_1^{(0,2)}(t) + \beta w_2^2 \pi_0(t)
\end{cases}
$$

$$
\begin{cases}
\dot{\mu}_0^{(1,1)}(t) = -4\mu_0^{(1,1)}(t) - 2\mu_0^{(0,2)}(t) + \mu_0^{(2,0)}(t) - \beta\mu_0^{(1,1)}(t) + \alpha v_1 v_2 \pi_1(t) \\
\dot{\mu}_1^{(1,1)}(t) = -3\mu_1^{(1,1)}(t) - \alpha\mu_1^{(1,1)}(t) + \beta w_1 w_2 \pi_0(t)
\end{cases}
\quad (41)
$$

The solutions of the above differential equations are substituted in (32) to obtain the second-order moment of the accumulated reward. Following a similar procedure, other moments of interest can be computed. For instance, the expected value of the reward is given by $\mathbb{E}[Y(t)] = \sum_{i=0}^{1} r_1(i)\mu_i^{(1,0)}(t) + r_2(i)\mu_i^{(0,1)}(t)$. *To compute* $\mu_i^{(1,0)}(t)$ *and* $\mu_i^{(0,1)}(t)$, *we would substitute* $m = (1,0)$ *and* $m = (0,1)$ *in (40).*

Notice that (40) also yields the Chapman–Kolmogorov differential equations that govern the evolution of the occupational probabilities $\pi_0(t)$ *and* $\pi_1(t)$. *Towards this end, substituting* $m = (0,0)$ *in (40), we obtain*

$$
\begin{cases}
\dot{\pi}_0(t) = -\beta\pi_0(t) + \alpha\pi_1(t), \\
\dot{\pi}_1(t) = -\alpha\pi_1(t) + \beta\pi_0(t),
\end{cases}
\quad (42)
$$

which are precisely the Chapman–Kolmogorov differential equations for a two-state CTMC. As discussed in Section 3.2, the value of the continuous state does not affect the discrete state dynamics when the transition rates are constant.

For illustration, we chose the parameters $\alpha = 6\ s^{-1}$, $\beta = 4\ s^{-1}$, $v = [v_1, v_2]^T = [10, -3]^T$, *and* $w = [w_1, w_2]^T = [-10, 8]^T$. *Figure 2 plots the occupational probabilities* $\pi_0(t)$ *and* $\pi_1(t)$ *computed by simulating (42), and averaging the results of 2000 Monte Carlo simulations. Figures 3 and 4 plot the first- and second-order moments of the reward obtained from the SHS approach with the results of 2000 Monte Carlo simulations superimposed in each case. The experiment is performed on a PC with a 2.53 GHz Intel® Core™ i5 CPU processor with 4 GB memory in the MATLAB® environment. The computer execution time for the Monte Carlo simulations was 88.25 s, while the computer execution time to obtain the moments with the SHS approach was 0.053 s.* □
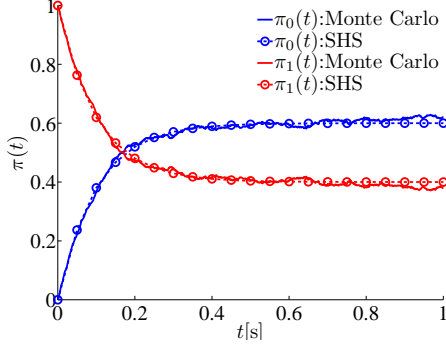
Figure 2: Occupational probabilities for the model studied in Example 1.
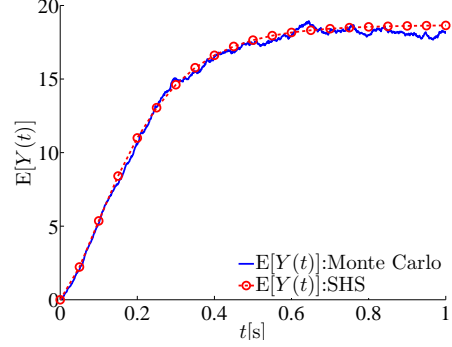

Figure 3: First-order moment of accumulated reward for the model studied in Example 1.
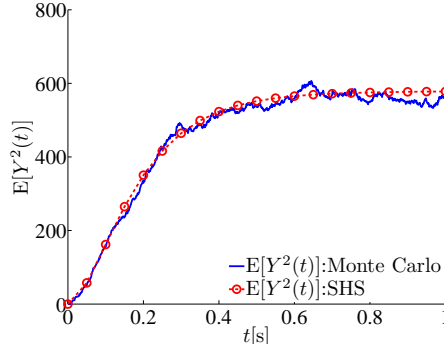

Figure 4: Second-order moment of the accumulated reward for the model studied in Example 1.

4. Case Studies

In this section, we present two numerical case studies to demonstrate the applicability of the proposed SHS-based framework in modeling the performability of dependable systems. To demonstrate the validity of the proposed approach, we compare the accuracy of the SHS modeling framework with Monte Carlo simulations and/or results from previous literature as appropriate.

The first case study examines the repair cost expended in maintaining a system of two electric-power transformers. The system is cast as a rate-reward model with impulses in the cost (associated with the one-time expense of enlisting the services of a repair crew). Inflationary effects are modeled with a discount rate. This model is adopted from [1], where the first-order moment of the accumulated repair cost was derived using a method based on the frequency of transitions of the underlying CTMC. We develop an SHS-based reward model for this system, and reproduce the results in [1]. In addition, we also obtain higher-order moments of the accumulated reward. In the second case study, we consider a second-order reward model that was introduced in [23] to describe the performance of a communication network. A Laplace-transform based method was adopted in [23] to obtain the moments of the accumulated reward. We reproduce the
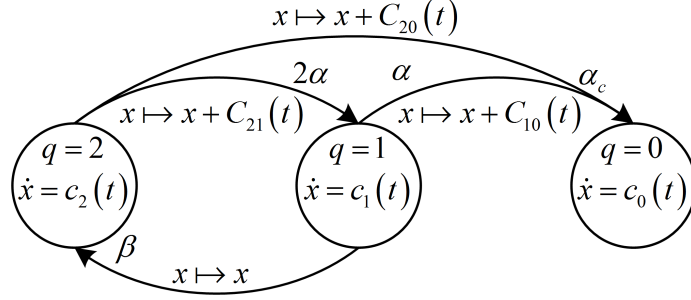
Figure 5: Markov reward model for a system of two transformers with common-cause failures.

results in [23] using the SHS modeling framework, and in addition, consider cases where there are losses and impulses in the accumulated reward.

4.1. Rate Reward Model with Impulses

This case study demonstrates how the SHS-based framework can be applied to model impulses in a rate-reward model. We examine the accumulated repair cost to maintain a system of two electric-power transformers with common-cause failures [1]. The state-transition diagram that describes the reliability of the system is depicted in Fig 5. The CTMC that describes the Markov reliability model is denoted by $Q(t) \in \mathcal{Q} = \{0, 1, 2\}$. In operational mode 2, both transformers are functioning, in operational mode 1, a single transformer is functioning, and in operational mode 0, both transformers have failed. The failure rate, repair rate, and common-cause failure rate are denoted by α, β, and α_c, respectively. The reward of interest is the cost of repair, denoted by $X(t)$. The rate at which the repair cost grows in the operational mode corresponding to discrete state i is denoted by $c_i(t)$ [\$/yr]. Transitions due to failures are associated with impulses in the repair cost that model the one-time expenses in enlisting the services of a repair crew. The impulse change in repair cost as a result of a failure transition from operational mode i to mode j is denoted by $C_{ij}(t)$ [\$]. The cost parameters are modeled to be time-dependent to factor inflation. In particular, we presume—following along the model in [1]—that $c_i(t) = c_i e^{-\gamma t}$ and $C_{ij}(t) = C_{ij} e^{-\gamma t}$. The parameter γ is the discount rate that represents future costs by a discounted value [1]. The authors in [1] obtain analytical expressions for the expected value of the accumulated repair cost with a method that is based on the frequency of visits in a CTMC [32]. We demonstrate how to cast this problem in the SHS-based framework. In doing so, we obtain a family of ODEs whose solution not only yields the expected value of the accumulated cost, but also higher-order moments (the higher order moments were not tackled in [1]).

We begin by defining test functions for each state of the CTMC:

$$\psi_i^{(m)}(q,\, x) = \delta_{q,i} x^m = \begin{cases} x^m & \text{if} \quad q = i \\ 0 & \text{if} \quad q \neq i \end{cases}, \, i \in \mathcal{Q} = \{0, 1, 2\}. \tag{43}$$

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If it turns out that there is only one transition between any two modes in the system, we can considerably simplify the definition of the SHS. In particular, we assume that the reset maps are of the form

$$\phi_j(q, x) = (j, \chi_j(q, x)), \quad \chi_j(q, x) = \sum_{j' \in \mathcal{J}} \delta_{q,j'} \omega_{j,j'}(x), \quad \forall j, j' \in \mathcal{J}, \tag{44}$$

where $\mathcal{J}$ is the set of transitions in the reward model. In particular, this means that the reset for the transition from operational mode j to j' is given by $\omega_{j,j'}(x)$. We then have

$$\psi_i^{(m)}(\phi_j(q, x)) = \psi_i^{(m)} \left(j, \sum_{j' \in \mathcal{J}} \delta_{q,j'} \omega_{j,j'}(x) \right) = \delta_{i,j} \left(\sum_{j' \in \mathcal{J}} \delta_{q,j'} \omega_{j,j'}(x) \right)^m$$

$$= \delta_{i,j} \sum_{j' \in \mathcal{J}} \delta_{q,j'} (\omega_{j,j'}(x))^m, \tag{45}$$

and therefore

$$\sum_{j \in \mathcal{J}} \lambda_j(q, x) \left(\psi_i^{(m)}(\phi_j(q, x)) - \psi_i^{(m)}(q, x) \right) = \lambda_j(q, x) \left(\sum_{j' \in \mathcal{J}} \delta_{q,j'} (\omega_{j,j'}(x))^m - \psi_i^{(m)}(q, x) \right). \tag{46}$$

To perform this computation, notice that we have to define

$$\lambda_0(q, x) = \delta_{q,1}\alpha + \delta_{q,2}\alpha_c, \quad \lambda_1(q, x) = \delta_{q,2}2\alpha, \quad \lambda_2(q, x) = \delta_{q,1}\beta, \tag{47}$$

and

$$\omega_{1,0}(x) = x + C_{10}(t), \quad \omega_{2,1}(x) = x + C_{21}(t), \quad \omega_{2,0}(x) = x + C_{20}(t), \quad \omega_{1,2}(x) = x. \tag{48}$$

The extended generators are given by

$$\left(L\psi_0^{(m)} \right)(q, x) = mc_0(t)\psi_0^{(m-1)}(q, x) + \alpha \left(\psi_1^{(1)}(q, x) + C_{10}(t)\psi_1^{(0)}(q, x) \right)^m$$
$$+ \alpha_c \left(\psi_2^{(1)}(q, x) + C_{20}(t)\psi_2^{(0)}(q, x) \right)^m, \tag{49}$$

$$\left(L\psi_1^{(m)} \right)(q, x) = mc_1(t)\psi_1^{(m-1)}(q, x) - (\alpha + \beta)\psi_1^{(m)}(q, x)$$
$$+ 2\alpha \left(\psi_2^{(1)}(q, x) + C_{21}(t)\psi_2^{(0)}(q, x) \right)^m, \tag{50}$$

$$\left(L\psi_2^{(m)} \right)(q, x) = mc_2(t)\psi_2^{(m-1)}(q, x) - (2\alpha + \alpha_c)\psi_2^{(m)}(q, x) + \beta\psi_1^{(m)}(q, x). \tag{51}$$

Applying (17) to (49)-(51), we obtain the following set of differential equations:

$$\frac{d}{dt}\mu_0^{(m)}(t) = mc_0(t)\mu_0^{(m-1)}(t) + \alpha \left(C_{10}^m(t)\pi_1(t) + \sum_{k=0}^{m-1} \binom{m}{k} \mu_1^{(m-k)}(t)C_{10}^k(t) \right)$$
$$+ \alpha_c \left(C_{20}^m(t)\pi_2(t) + \sum_{k=0}^{m-1} \binom{m}{k} \mu_2^{(m-k)}(t)C_{20}^k(t) \right), \tag{52}$$

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$$\frac{d}{dt}\mu_1^{(m)}(t) = mc_1(t)\mu_1^{(m-1)}(t) - (\alpha+\beta)\,\mu_1^{(m)}(t)$$
$$+ 2\alpha\left(C_{21}^m(t)\pi_2(t) + \sum_{k=0}^{m-1}\binom{m}{k}\mu_2^{(m-k)}(t)C_{21}^k(t)\right), \tag{53}$$

$$\frac{d}{dt}\mu_2^{(m)}(t) = mc_2(t)\mu_2^{(m-1)}(t) - (2\alpha+\alpha_c)\,\mu_2^{(m)}(t) + \beta\mu_1^{(m)}(t), \tag{54}$$

where $\pi_0(t)$, $\pi_1(t)$, and $\pi_2(t)$ are the occupational probabilities of the different modes. Notice that substituting $m=0$ in (52)-(54) recovers the Chapman-Kolmogorov equations: $\dot{\pi}(t) = \pi(t)\Lambda$, where $\pi(t) = [\pi_0(t),\,\pi_1(t),\,\pi_2(t)]$, and Λ is given by:

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & -(\alpha+\beta) & \beta \\ \alpha_c & 2\alpha & -(2\alpha+\alpha_c) \end{bmatrix}.$$

The m-order moment of the accumulated repair cost is given by (31), i.e., $\mathbb{E}[X^m(t)] = \mu_0^{(m)}(t) + \mu_1^{(m)}(t) + \mu_2^{(m)}(t)$. The evolution of $\mu_0^{(m)}(t)$, $\mu_1^{(m)}(t)$, and $\mu_2^{(m)}(t)$ is given by (52)-(54).

For illustration, consider: $\alpha = 2\,\mathrm{yr}^{-1}$, $\beta = 1000\,\mathrm{yr}^{-1}$, $\alpha_c = 1\,\mathrm{yr}^{-1}$, $c_2 = 1000\,\$/\mathrm{yr}$, $c_1 = 10,000\,\$/\mathrm{yr}$, $c_0 = 0\,\$$, $C_{21} = 500\,\$$, $C_{20} = 1000\,\$$, and $C_{10} = 500\,\$$ [1]. Figure 6 depicts the expected value of the accumulated repair cost for two different values of γ. The results from the SHS approach (obtained by simulating (52)-(54) for $m=1$, and then using $\mathbb{E}[X(t)] = \mu_0^{(1)}(t) + \mu_1^{(1)}(t) + \mu_2^{(1)}(t)$) are superimposed to the results from [1]. To further validate the approach, Figs. 7-8 depict the second- and third-order moments of the accumulated cost (obtained by simulating (52)–(54) for $m=2$ and $m=3$, respectively) superimposed to results obtained from 5000 Monte Carlo simulations. Note that it is unclear how the method proposed in [1] can be extended to obtain higher-order moments. Therefore, in these cases, we just include the Monte Carlo results for comparison and validation of the SHS approach.
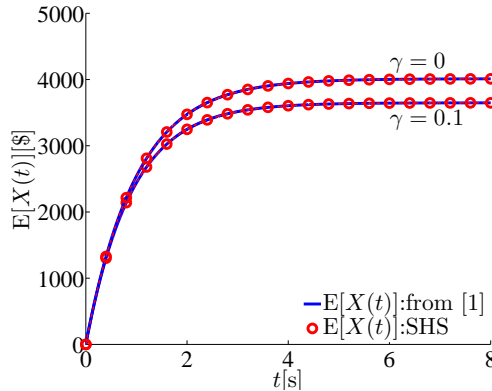

Figure 6: Expected value of accumulated repair cost for the transformer reliability model.
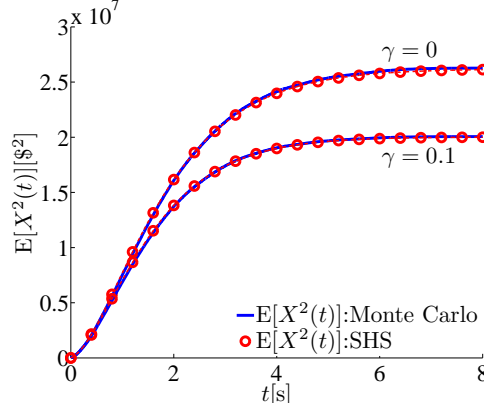
18

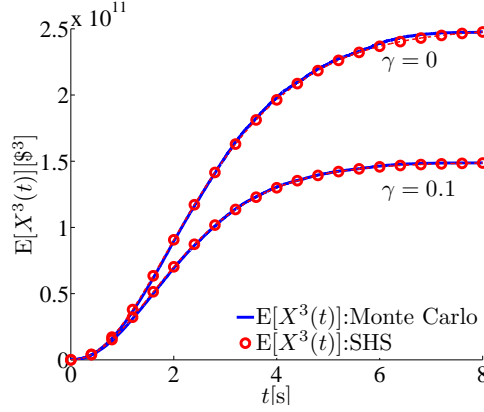Figure 7: Second-order moment of accumulated repair cost for the transformer reliability model.

Figure 8: Third-order moment of accumulated repair cost for the transformer reliability model.

4.2. Second-order Reward Model

In this case study, we examine the second-order Markov reward model illustrated by the state-transition diagram in Fig. 9. Note that this is a generalized version of the model presented in [23], which was employed to model the capacity of a communication channel (the reward is the available channel capacity). Transitions between different modes and the associated transition rates are also illustrated in the figure. We assume that failure transitions are associated with a reset map that can model partial total loss or impulses in the accumulated reward. In partial total loss models, a (possibly mode-dependent) fraction of the total accumulated reward is lost with each transition of the discrete state. With regard to the state-transition diagram presented in Fig. 9, setting $C_{ij} \equiv 0$, $0 \leq \kappa_i \leq 1$, we recover a model that captures partial total loss in the accumulated reward. Similarly, choosing $C_{ij} < 0$, $\kappa_i \equiv 0$, models impulses in the accumulated reward. The moments of the accumulated reward are derived from a direct analysis of the Laplace transform of the accumulated-reward probability distribution in [23]. Here, we demonstrate how to formulate the model within the SHS-based framework. As before, begin by defining the following test functions:

$$\psi_i^{(m)}(q,\,x) = \begin{cases} x^m & \text{if} \quad q = i \\ 0 & \text{if} \quad q \neq i \end{cases} \,,\ i = 0,\,1,\,\ldots,\,N. \tag{55}$$

The generators for the states can be obtained from (16) as follows:

$$\begin{aligned} \left(L\psi_0^{(m)}\right)(q,\,x) &= ma_0\psi_0^{(m)}(q,\,x) + mb_0\psi_0^{(m-1)}(q,\,x) - N\beta\psi_0^{(m)}(q,\,x) \\ &\quad + \alpha\left(\kappa_1\psi_1^{(1)}(q,\,x) + C_{10}\right)^m \psi_1^{(0)}(q,\,x) + \frac{1}{2}\sigma_0^2 m(m-1)\psi_n^{(m-2)}(q,\,x), \end{aligned} \tag{56}$$

$$\begin{aligned} \left(L\psi_N^{(m)}\right)(q,x) &= ma_N\psi_N^{(m)}(q,x) + mb_N\psi_N^{(m-1)}(q,x) + \beta\psi_{N-1}^{(m)}(q,x) \\ &\quad - N\alpha\psi_n^{(m)}(q,x) + \frac{1}{2}\sigma_N^2 m(m-1)\psi_N^{(m-2)}(q,x), \end{aligned} \tag{57}$$

$$\begin{aligned} \left(L\psi_i^{(m)}\right)(q,x) &= ma_i\psi_i^{(m)}(q,x) + mb_i\psi_i^{(m-1)}(q,x) - ((N-i)\beta + i\alpha)\psi_i^{(m)}(q,x) \\ &\quad + (i+1)\alpha\left(\kappa_{i+1}\psi_{i+1}^{(1)}(q,x) + C_{(i+1)i}\right)^m \psi_{i+1}^{(0)}(q,x) + (N-(i-1))\beta\psi_{i-1}^{(m)}(q,x) \\ &\quad + \frac{1}{2}\sigma_i^2 m(m-1)\psi_i^{(m-2)}(q,x),\, \forall i = 1,\ldots,\,N-1. \end{aligned} \tag{58}$$

Define the conditional moments $\mu_i^{(m)}(t) = \mathbb{E}\left[\psi_i^{(m)}(q,\,x)\right]$, $i = 0,\,1,\,\ldots,\,N$, and denote the vector of conditional moments at time t by $\mu^{(m)}(t) = [\mu_0^{(m)}(t),\,\mu_1^{(m)}(t),\,\ldots,\,\mu_N^{(m)}(t)]$. Applying (17), we see that the evolution of $\mu_i^{(m)}(t)$, $i = 0,\,1,\,\ldots,\,N$ is governed by

$$\begin{aligned} \frac{d}{dt}\mu_0^{(m)}(t) &= ma_0\mu_0^{(m)}(t) + mb_0\mu_0^{(m-1)}(t) - N\beta\mu_0^{(m)}(t) + \frac{1}{2}\sigma_0^2 m(m-1)\mu_0^{(m-2)}(t) \\ &\quad + \alpha\left(C_{10}^m\pi_1(t) + \sum_{k=0}^{m-1}\binom{m}{k}\kappa_1^{m-k}\mu_1^{(m-k)}(t)C_{10}^k\right), \end{aligned} \tag{59}$$

$$\frac{d}{dt}\mu_N^{(m)}(t) = ma_N\mu_N^{(m)}(t) + mb_N\mu_N^{(m-1)}(t) + \beta\mu_{N-1}^{(m)}(t) - N\alpha\mu_N^{(m)}(t) + \frac{1}{2}\sigma_N^2 m(m-1)\mu_N^{(m-2)}(t), \tag{60}$$
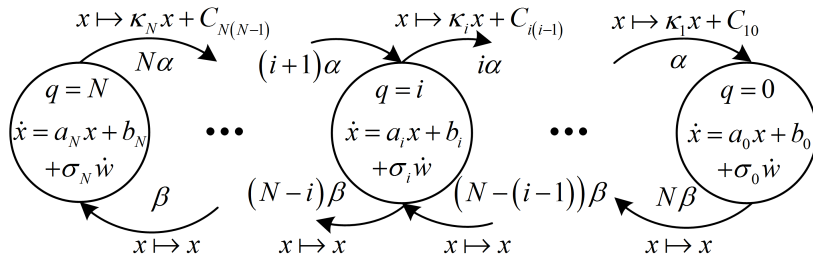

Figure 9: State-transition diagram for the second-order reward model.

$$
\begin{aligned}
\frac{d}{dt}\mu_i^{(m)}(t) =\ & ma_i\mu_i^{(m)}(t) + mb_i\mu_i^{(m-1)}(t) - ((N-i)\beta + i\alpha)\mu_i^{(m)}(t) \\
& + (i+1)\alpha\left(C_{(i+1)i}^m\pi_{i+1}(t) + \sum_{k=0}^{m-1}\binom{m}{k}\kappa_{i+1}^{m-k}\mu_{i+1}^{(m-k)}(t)C_{(i+1)i}^k\right) \\
& + (N-(i-1))\beta\mu_{i-1}^{(m)}(t) + \frac{1}{2}\sigma_i^2 m(m-1)\mu_i^{(m-2)}(t),\ \forall i = 1,\ldots,N-1. \quad (61)
\end{aligned}
$$

As a special case, consider $a_i = 0$, $b_i = (C - ir)$, $\sigma_i = \sqrt{i}\sigma$, $\kappa_i = 1$, and $C_{ij} = 0$. This recovers the model studied in [23], where there are no losses in the accumulated reward. In this case, (59)–(61) simplify to the following

$$
\frac{d}{dt}\mu^{(m)}(t) = \mu^{(m)}(t)\Lambda + m\mu^{(m-1)}(t)\Gamma + \frac{1}{2}m(m-1)\mu^{(m-2)}(t)\Upsilon, \quad (62)
$$

where $\mu^{(m)}(t)$ is the vector of conditional moments at time t,

$$
\Gamma = \mathrm{diag}\left(C,\ldots C - ir,\ldots,C - Nr\right), \quad (63)
$$

$$
\Upsilon = \mathrm{diag}\left(0,\ldots i\sigma^2,\ldots,N\sigma^2\right), \quad (64)
$$

$$
\Lambda = \begin{bmatrix}
-N\beta & N\beta & 0 & 0 & 0 & \ldots & 0 \\
\alpha & * & (N-1)\beta) & 0 & 0 & \ldots & 0 \\
0 & 2\alpha & * & (N-2)\beta & 0 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & 0 & 0 & \ldots & (N-1)\alpha & * & \beta \\
0 & 0 & 0 & 0 & \ldots & N\alpha & -N\alpha
\end{bmatrix}. \quad (65)
$$

To save space, we have sometimes written the diagonal elements of this matrix by a $*$, but of course it is implied by the fact that Λ must be zero-sum, e.g., the $(2,2)$ element of the matrix is $-(\alpha + (N-1)\beta)$, etc. Also notice that Λ is the generator matrix of the underlying CTMC. The expression in (62) exactly matches Equation (6) in Theorem 2 of [23].

For illustration, consider the following: $N = 10$, $\alpha = 5$, $\beta = 2$, $\kappa_i = 0.5$, $C_{ij} = -0.1$, $a_i = i$, $b_i = N$, $\sigma_i = \sqrt{i}\sigma$. Figures 10, 11, 12 plot the first-, second-, and third-order moments of the reward obtained from the SHS approach (substituting $m = 1, 2, 3$, respectively in (59)-(61), and using (31)). The results of 75,000 Monte Carlo simulations are superimposed in each case; simulations are repeated for different values of σ to demonstrate the validity of the approach.

We note that the Monte Carlo approximation for high moments becomes quite *intermittent*, especially when σ is large (here, we roughly mean that the trajectory has many "spikes"). This is a predictable feature of the system; when we are computing high moments, a single large realization can make a very significant change to the empirically measured moment. Of course, in the limit of taking infinitely many samples, this effect dies out, but notice that for high moments we would need to take a very large number of samples and thus the method we propose here becomes even more preferable for higher-order moments.
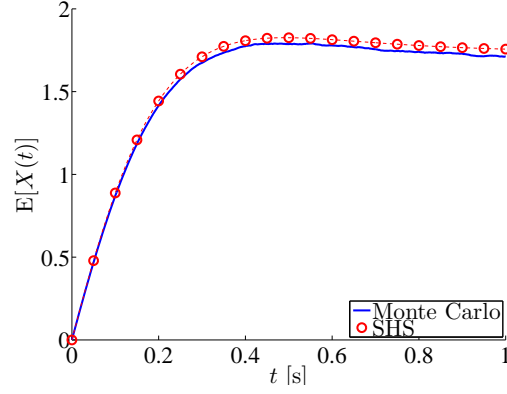
Figure 10: Expected value of the accumulated reward for the second-order reward model.
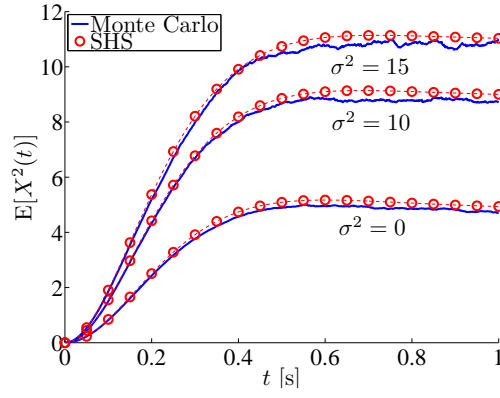

Figure 11: Second-order moment of the accumulated reward for the second-order reward model.
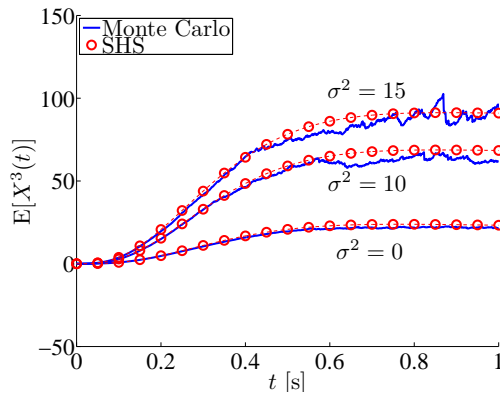

Figure 12: Third-order moment of the accumulated reward for the second-order reward model.
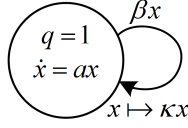
22

Figure 13: Single discrete-state SHS with continuous-state-dependent transition rate.

5. The Problem of Moment Closure in Markov Reward Models

Recall that in the class of reward models explored so far, the vector fields that govern the evolution of the continuous state and the reset maps are linear, while the transition rates are independent of the continuous state. If these assumptions are relaxed, the differential equations that govern the evolution of the moments are infinite dimensional and *moment-closure techniques* have to be applied to solve them.

To see the added difficulty, notice that in all of the cases considered previously in this paper, the evolution equation for the pth order moments of the process have always depended on lower-order moments, and thus, the moment evolution equations always give a closed system. For example, we could always first solve the Chapman–Kolmogorov equations to obtain the zeroth-order moments; from this, the equations for the first-order moments depended only on themselves and these zeroth-order moments, the second-order moments only depend on themselves and lower order, etc. In general, however, we run into may a case where the evolution equation for a moment of a given order depends on higher-order moments; the resulting system is not closed and cannot be solved. We illustrate this next with a simple example.

Example 4. *Consider the state transition diagram illustrated in Fig. 13, for a first-order reward model with a single discrete state. The transition rate and the reset map are both linear functions of the continuous state in this case. The generator for this process is given by*

$$
\begin{aligned}
\left(L\psi^{(m)}\right)(x) &= \frac{\partial}{\partial x}\psi^{(m)}(x) \cdot ax + \beta x \left(\psi^{(m)}\left(\phi(x)\right) - \psi^{(m)}\left(x\right)\right) \\
&= ma\psi^{(m)}(x) + \beta\left(\kappa^m - 1\right)\psi^{(m+1)}(x).
\end{aligned}
\tag{66}
$$

Applying (17), we see that the evolution of the moments of $X(t)$ is governed by

$$
\dot{\mu}^{(m)}(t) = ma\mu^{(m)}(t) + \beta\left(\kappa^m - 1\right)\mu^{(m+1)}(t).
\tag{67}
$$

Notice that $\dot{\mu}^{(m)}(t)$ depends on $\mu^{(m+1)}(t)$. Therefore, moment-closure methods are required to solve (67), i.e., to simulate the differential equation that governs the m-order moment, $\mu^{(m+1)}(t)$ has to be expressed as some function of $\mu^{(i)}(t)$, $1 \leq i \leq m$. □

Typically, moment-closure methods rely on assumptions about the underlying probability distribution of the state. Methods tailored to SHS are described in [20, 21] and the references therein. For the reward models introduced in Section 3.2, moment-closure methods are unnecessary—as demonstrated in the case studies, this class of reward models is still very powerful and can be applied to a variety of system performability modeling problems. A detailed discussion of moment-closure methods (as they apply to reward models with continuous-state-dependent transition rates and/or general polynomial vector fields governing the continuous states) is beyond the scope of this work and part of ongoing research.

6. Concluding Remarks

This work presented a unified framework to analyze Markov reward models based on the theory of SHS. The moments of the accumulated reward are obtained by the solution of ODEs that govern the conditional moments of the accumulated reward. The framework provides a unified solution approach to rate, first-order, and second-order reward models with impulses and/or losses. Additionally, it is computationally inexpensive (by orders of magnitude in some cases) compared to Monte Carlo simulations. Future work includes analyzing reward models governed by nonlinear SDEs, with transition rates that are a function of time and/or the accumulated reward (a primer to this problem was given in Section 5).

AppendixA. Rigorous Definition of SHS

In Section 3.1, we gave an asymptotic and intuitive description of an SHS. Here we give the precise definition of an SHS as a stochastic process, and demonstrate that it does have the same asymptotics as described in Section 3.1. Let us consider the case where the continuous-state dynamics are deterministic, i.e., $g(q, x, t) \equiv 0$ in (9), or, said another way, the continuous-state dynamics are governed by an ODE and not an SDE:[6]

$$\frac{d}{dt} X(t) = f(Q(t), X(t), t). \tag{A.1}$$

Denote the *flow-map* of (A.1) by the function ξ, and define it as

$$\frac{d}{dt} \xi_{t_0}^t(q, x_0) = f(q, \xi_{t_0}^t(q, x_0), t), \quad \xi_{t_0}^{t_0}(q, x_0) = x_0. \tag{A.2}$$

The function $\xi_{t_0}^t(q, x_0)$ is the solution to the flow (A.1) at time t with q held fixed whenever the flow was started with value x_0 at time t_0.

Recall that we denote the set of transitions by $\mathcal{J}$, and the transition rates and reset maps by $\lambda_j \colon \mathcal{Q} \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^+$, $\phi_j \colon \mathcal{Q} \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathcal{Q} \times \mathbb{R}^d$, $j \in \mathcal{J}$. Let us now define the family of independent exponential random variables with rate one Z_j^k, where $j \in \mathcal{J}$ and $k \in \mathbb{N}$ indexes the jump number, such that $Z_j^k \geq 0$, and

$$\Pr\{Z_j^k > t\} = e^{-t} \text{ for } t > 0. \tag{A.3}$$

We then define a family of stopping times recursively; it is at these stopping times that the value of the discrete state will change. More precisely, let us define $T_0 = 0$, and

$$T_1 = \min_{t>0}\{\exists j \text{ with } \int_0^t \lambda_j(Q(0), \xi_{T_0}^s(Q(0), X(0)), s) \, ds \geq Z_j^1\}, \tag{A.4}$$

---

and define J_1 to be the j at which the integral crosses at time T_1. Notice that J_1 is uniquely determined with probability one, since the probability of two independent exponentials being equal is zero under any smooth change of time coordinate (this is a standard result in stochastic processes, see, e.g., [4, 34]).

With T_1 and J_1 in hand, we can now define $Q(t), X(t)$ for $t \in [0, T_1]$ as follows: we define, for all $t \in [0, T_1)$,

$$Q(t) = Q(0), \quad X(t) = \xi_{T_0}^t(Q(0), X(0)), \tag{A.5}$$

and at $t = T_1$, we define

$$(Q(T_1), X(T_1)) = \phi_{J_1}(Q(T_1^-), X(T_1^-), T_1). \tag{A.6}$$

In short, we require that no jump occur until time T_1, and until this time we hold the discrete state constant and flow the continuous state according to the appropriate ODE; at time T_1 we enforce jump j to occur. Notice that the limit in (A.6) must exist, since $Q(t), X(t)$, as defined in (A.5) are continuous functions on $[0, T_1]$—of course $Q(t)$ is constant and thus continuous, and $X(t)$ is the solution of an ODE with a Lipschitz vector field and is, in fact, continuously differentiable.

This defines the process only until the time of the first jump, but then we can extend the argument recursively. Say that we know the value of the process $(Q(t), X(t))$ on $[0, T_m]$, then we define T_{m+1}, J_{m+1} as follows:

$$T_{m+1} = \min_{t > T_m} \{ \exists j \text{ with } \int_{T_m}^t \lambda_j(Q(T_m), \xi_{T_m}^s(Q(T_m), X(T_m)), s) \, ds \geq Z_j^{m+1} \}, \tag{A.7}$$

and J_{m+1} is the index at which this occurs. We then define, for all $t \in [T_m, T_{m+1})$,

$$Q(t) = Q(T_m), \quad X(t) = \xi_{T_m}^t(Q(T_m), X(T_m)), \tag{A.8}$$

and at $t = T_{m+1}$, we define

$$(Q(T_{m+1}), X(T_{m+1})) = \phi_{J_{m+1}}(Q(T_m^-), X(T_m^-), T_m). \tag{A.9}$$

Of course, we want to verify that this definition is consistent with the asymptotic notions presented in Section 3.1. So let us consider the event that we have observed exactly m transitions at time t, that the $(m + 1)^{\text{st}}$ transition occurs in $(t, t + \Delta t]$, and that it is the jth transition that occurs, i.e., compute the probability

$$\Pr\{(T_{m+1} < t + \Delta t) \wedge (J_{m+1} = j)|T_{m+1} > t\}. \tag{A.10}$$

Defining $\zeta_j(t) = \int_0^t \lambda_j(Q(T_m), \xi_{T_m}^s(Q(T_m), X(T_m)), s) \, ds$, we see that

$$\zeta_j(t + \Delta t) - \zeta_j(t) = \int_t^{t+\Delta t} \lambda_j(Q(T_m), \xi_{T_m}^s(Q(T_m), X(T_m)), s) \, ds$$

$$= \Delta t \cdot \lambda_j(Q(T_m), \xi_{T_m}^t(Q(T_m), X(T_m)), T_m) + o(\Delta t). \tag{A.11}$$

However, if T is an exponential random variable with rate one, then for $t < s$,

$$\Pr\{T < s | T > t\} = \Pr\{T < s - t\} = 1 - e^{-(s-t)} = (s - t) + O((s - t)^2), \tag{A.12}$$

and therefore

$$\Pr\{(T_{m+1} < t + \Delta t) \wedge (J_{m+1} = j) | T_{m+1} > t\} = \Delta t \cdot \lambda_j(Q(T_m), \xi_{T_m}^t(Q(T_m), X(T_m)), T_m) + o(\Delta t)$$

$$= \Delta t \cdot \lambda_j(Q(t), X(t), t) + o(\Delta t), \tag{A.13}$$

agreeing with the definition in Section 3.1.

AppendixB. Justification of Dynkin's Formula: Operator Extended Domain

Let us define $N(t)$ to be the number of jumps the process has taken at time t, i.e.,

$$N(t) = k \iff T_k \leq t \wedge T_{k+1} > t. \tag{B.1}$$

Following Theorem 26.14 of [8], any function $\psi \colon \mathcal{Q} \times \mathbb{R}^d \to \mathbb{R}$ is in the (extended) domain of the operator L (specifically, this means that, as given in (17), Dynkin's formula holds for ψ) if the conditions

$$\mathbb{E}_{q_0, x_0}[N(t)] < \infty, \forall t > 0, q_0 \in \mathcal{Q}, x_0 \in \mathbb{R}^d, \tag{B.2}$$

and

$$\mathbb{E}_{q_0, x_0}\left[\sum_{T_n < t} |\psi(Q(T_n), X(T_n)) - \psi(Q(T_n^-), X(T_n^-))|\right] < \infty, \forall t > 0, q_0 \in \mathcal{Q}, x_0 \in \mathbb{R}^d \tag{B.3}$$

hold. First, notice that as long as we have bounded rates, then (B.2) will hold. More specifically, if there exists $\overline{\lambda}$ such that $\lambda(q, x, t) \leq \overline{\lambda}$ for all $q \in \mathcal{Q}, x \in \mathbb{R}^d$, then $\mathbb{E}[N(t)] \leq t / \overline{\lambda}$ (this is a standard result, see, e.g., Theorem 2.3.2 in [34]). If we further assume that ψ is uniformly bounded, i.e., that there exists $\overline{\psi}$ such that $\psi(q, x) \leq \overline{\psi}$ for all $q \in \mathcal{Q}, x \in \mathbb{R}^d$, then (B.2) implies (B.3). So, in short, uniform bounds on λ and ψ guarantee (B.2) and (B.3) and this implies that (17) holds for ψ.

However, we are particularly interested in applying Dynkin's formula for *unbounded* ψ, e.g., functions that are polynomial in the argument. Recall from (18) that these are the type of test functions we use. If we weaken the assumption of bounded ψ to allow for continuous ψ, but then assume that there exists a function $\alpha(t)$ with $\alpha(t) < \infty$ for all $t > 0$, such that $|X(t)| < \alpha(t)$, then again (B.2) implies (B.3). It is

not hard to show, using standard dynamical systems techniques, that this holds for SHS where the flows describing the evolution of the continuous state are governed by (10) and the reset maps are bounded.

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