

# Duality, Achievable Rates, and Sum-Rate Capacity of Gaussian MIMO Broadcast Channels

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**Abstract**—We consider a multiuser multiple-input multiple-output (MIMO) Gaussian broadcast channel (BC), where the transmitter and receivers have multiple antennas. Since the MIMO BC is in general a nondegraded BC, its capacity region remains an unsolved problem. In this paper, we establish a duality between what is termed the “dirty paper” achievable region (the Caire–Shamai achievable region) for the MIMO BC and the capacity region of the MIMO multiple-access channel (MAC), which is easy to compute. Using this duality, we greatly reduce the computational complexity required for obtaining the dirty paper achievable region for the MIMO BC. We also show that the dirty paper achievable region achieves the sum-rate capacity of the MIMO BC by establishing that the maximum sum rate of this region equals an upper bound on the sum rate of the MIMO BC.

**Index Terms**—Broadcast channel (BC), channel capacity, dirty paper coding, duality, multiple-input multiple-output (MIMO) systems.

## I. INTRODUCTION

MULTIPLE-input multiple-output (MIMO) systems have received a great deal of attention as a method to achieve very high data rates over wireless links. The capacity of single-user MIMO Gaussian channels was first studied by Foschini [1] and Telatar [2]. This work has also been extended to the MIMO multiple-access channel (MAC) [2]–[4]. The capacity of MIMO broadcast channels (BC), however, is an open problem due to the lack of a general theory on nondegraded BCs. In pioneering work by Caire and Shamai [5], a set of achievable rates (the achievable region) for the MIMO BC was obtained by applying the “dirty paper” result [6] at the transmitter (or alternatively, coding for noncausally known interference). It was also shown in [5], [7], that the sum rate MIMO BC capacity equals the maximum sum rate of this achievable region for the two-user BC with an arbitrary number of transmit antennas ( $t \geq 1$ ) and one receive antenna at each receiver ( $r_1 = r_2 = 1$ ). However, computing this region is extremely complex and the approach used in [5], to prove the optimality of dirty paper coding for sum rate does not appear to work for the more general class of channels (i.e., an arbitrary number of users and receive antennas) which we consider.

In this paper, we consider a  $K$ -user MIMO Gaussian BC in which receiver  $j$  has  $r_j \geq 1$  receive antennas and the transmitter has  $t \geq 1$  transmit antennas. The achievable region for a general MIMO BC requires an extension of the Caire–Shamai region to multiple users and multiple receive antennas, which was done by Yu and Cioffi in [8]. We refer to this extension as the dirty paper region. We establish a duality between the dirty paper region of the MIMO BC and the capacity region of the MIMO MAC. In other words, we show that the dirty paper region is exactly equal to the capacity region of the dual MIMO MAC, with the  $K$  transmitters having the same *sum* power constraint as the MIMO BC. We establish this duality by showing that all rates achievable in the dual MIMO MAC with power constraints whose sum equals the BC power constraint are also achievable in the MIMO BC, and *vice versa*. This duality is the multiple-antenna extension of the previously established duality between the scalar Gaussian BC and MAC [9]. Though we consider only the constant channel case, this duality can easily be shown to hold for fading multiple-antenna Gaussian BCs and MACs, as it does in the scalar channel case. This duality has two important applications: it permits us to compute with ease the dirty paper region of the MIMO BC, which is very difficult to compute directly, and it allows us to show that the dirty paper achievable region achieves the sum rate capacity of the MIMO BC.

Finding the full capacity region of the MIMO BC is very difficult due to its nondegraded nature, but we are able to show that the dirty paper region achieves the sum rate capacity of the MIMO BC through the use of the Sato upper bound on the sum-rate capacity of BCs [10]. The Sato upper bound was first applied to the MIMO BC by Caire and Shamai [5] to find the sum rate capacity of the ( $t \geq 1, r_1 = r_2 = 1$ ) channel. Using Sato’s technique, we bound the sum rate capacity of the MIMO BC by considering the capacity when the  $K$  receivers perform joint signal detection (i.e., we consider a single-user  $t \times (\sum_{j=1}^K r_j)$  antenna channel) with the worst case colored noise, when the noise at every antenna is correlated with the noise at every other antenna except for those at the same receiver, and we analytically show, employing a proof which generally constructs the worst case noise covariance matrix, that this upper bound coincides exactly with the maximum achievable sum rate in the dirty paper region.

There has been parallel work on the optimality of the dirty paper region for sum capacity of the BC. In [11], the authors handle those channels for which the worst case correlation of the noise in the Sato upper bound is nonsingular, and in [12], the authors handle channels with multiple transmit antennas but only one antenna at each receiver. In this paper, we deal with

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the most general case, with no restrictions on the number of antennas or on the worst case noise.

Although the optimality of the dirty paper region has only been shown for sum rate (and trivially for the corner points of the region), the fact that the dirty paper region is equal to the dual MIMO MAC capacity region together with the fact that the scalar Gaussian BC capacity region is equal to the dual MAC capacity region leads us to believe that the dirty paper region may actually be the capacity region of the MIMO BC. Significant progress toward proving this conjecture has recently been made [13], [14], but this hypothesis remains unproven.

The remainder of this paper is organized as follows. In Section II, we describe the MIMO BC and the dual MIMO MAC. In Section III, we summarize some background information, including the achievable ‘‘dirty paper’’ BC region, the MIMO MAC capacity region, and the duality of the scalar MAC and BC. We describe the MIMO MAC–BC duality result in Section IV. In Section V, we show that the dirty paper region achieves sum rate capacity of the MIMO BC and we provide a few illustrative examples in Section VI. We conclude with Section VII.

## II. SYSTEM MODEL

We use boldface to denote matrices and vectors.  $|\mathcal{S}|$  denotes the determinant and  $\mathcal{S}^{-1}$  the inverse of a square matrix  $\mathcal{S}$ . For any general matrix  $\mathbf{M}$ ,  $\mathbf{M}^\dagger$  denotes the conjugate transpose and  $\text{Tr}(\mathbf{M})$  denotes the trace.  $\mathbf{I}$  denotes the identity matrix and  $\text{diag}(\lambda_i)$  denotes a diagonal matrix with the  $(i, i)$  entry equal to  $\lambda_i$ .

We consider a MIMO BC with a  $t$ -antenna transmitter and  $K$  receivers with  $r_1, \dots, r_K$  receive antennas, respectively. The transmitter sends independent information to each receiver. The BC is the system on the left in Fig. 1.

Let  $\mathbf{x} \in \mathbb{C}^{t \times 1}$  be the transmitted vector signal and let  $\mathbf{H}_k \in \mathbb{C}^{r_k \times t}$  be the channel matrix of receiver  $k$  where  $\mathbf{H}_k(i, j)$  represents the channel gain from transmit antenna  $j$  to antenna  $i$  of receiver  $k$ . The white Gaussian noise at receiver  $k$  is represented by  $\mathbf{n}_k \in \mathbb{C}^{r_k \times 1}$  where  $\mathbf{n}_k \sim N(0, \mathbf{I})$ . Let  $\mathbf{y}_k \in \mathbb{C}^{r_k \times 1}$  be the received signal at receiver  $k$ . The received signal is mathematically represented as

$$\begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_K \end{bmatrix} = \mathbf{H}\mathbf{x} + \begin{bmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_K \end{bmatrix}, \quad \text{where } \mathbf{H} = \begin{bmatrix} \mathbf{H}_1 \\ \vdots \\ \mathbf{H}_K \end{bmatrix}. \quad (1)$$

The matrix  $\mathbf{H}$  represents the channel gains of all receivers. The covariance matrix of the input signal is  $\Sigma_{\mathbf{m}a_x} \triangleq \mathbb{E}[\mathbf{x}\mathbf{x}^\dagger]$ . The transmitter is subject to an average power constraint  $P$ , which implies  $\text{Tr}(\Sigma_{\mathbf{m}a_x}) \leq P$ . We assume the channel matrix  $\mathbf{H}$  is constant and is known perfectly at the transmitter and at all receivers.

Now consider the dual MAC shown in the right half of Fig. 1. The dual channel is arrived at by converting the receivers in the BC into transmitters in the MAC and converting the  $t$ -antenna transmitter into a  $t$ -antenna receiver. Notice that the channel gains of the dual MAC are the same as that of the BC, i.e.,  $\mathbf{H}_k(i, j)$  corresponds to the gain from transmit antenna  $j$  to antenna  $i$  of receiver  $k$  in the BC and to the gain from antenna  $i$  of transmitter  $k$  to receive antenna  $j$  in the MAC.

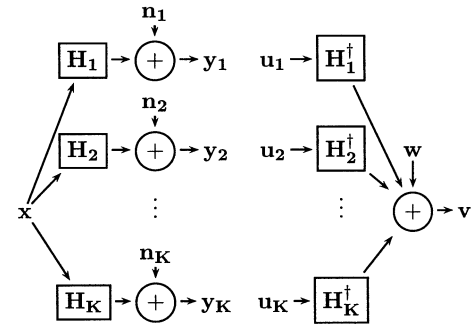


Fig. 1. System models of the MIMO BC (left) and the MIMO MAC (right) channels.

Let  $\mathbf{u}_k \in \mathbb{C}^{r_k \times 1}$  be the transmitted signal of transmitter  $k$ . Let  $\mathbf{v} \in \mathbb{C}^{t \times 1}$  be the received signal and  $\mathbf{w} \in \mathbb{C}^{t \times 1}$  the noise vector where  $\mathbf{w} \sim N(0, \mathbf{I})$ . The received signal is mathematically represented as

$$\begin{aligned} \mathbf{v} &= \mathbf{H}_1^\dagger \mathbf{u}_1 + \dots + \mathbf{H}_K^\dagger \mathbf{u}_K + \mathbf{w} \\ &= \mathbf{H}^\dagger \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_K \end{bmatrix} + \mathbf{w}, \quad \text{where } \mathbf{H}^\dagger = \begin{bmatrix} \mathbf{H}_1^\dagger & \dots & \mathbf{H}_K^\dagger \end{bmatrix}. \end{aligned}$$

In the dual MAC, each transmitter is subject to an individual power constraint of  $P_1, \dots, P_K$ , with  $\sum_{i=1}^K P_i = P$  (i.e., the sum of the MAC power constraints equals the BC power constraint). We also assume perfect knowledge of the channel at the transmitters and the receiver in the dual MAC.

Finally, we define the *cooperative system* to be the same as the BC, but with all receivers coordinating to perform joint detection. If the receivers are allowed to cooperate, the BC reduces to a single-user  $t \times (\sum_{j=1}^K r_j)$  multiple-antenna system described by

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z} \quad (2)$$

where

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_K \end{bmatrix} \quad \text{and} \quad \mathbf{z} = \begin{bmatrix} \mathbf{n}_1 \\ \vdots \\ \mathbf{n}_K \end{bmatrix}.$$

We call the capacity of this system the *cooperative capacity*.

We use

$$\mathcal{C}_{\text{BC}}(P, \mathbf{H}), \mathcal{C}_{\text{MAC}}(P_1, \dots, P_K, \mathbf{H}^\dagger), \text{ and } \mathcal{C}_{\text{coop}}(P, \mathbf{H})$$

to denote the capacity regions of the MIMO BC, MIMO MAC, and cooperative system, respectively.

## III. BACKGROUND

To obtain our results, we use the achievable region of the MIMO BC channel obtained in [5], [8] and results on the MIMO MAC capacity region [2]–[4], extensively in this paper. Hence, we first summarize these results and then state results on the duality of the scalar Gaussian BC and MAC [9].

### A. Achievable BC Region—The Dirty Paper Region

An achievable region for the MIMO BC was first obtained in [5]. In [8], the region was extended to the more general multiple-user, multiple-antenna case using the following extension of dirty paper coding [6] to the vector case.

*Lemma 1 (Yu, Cioffi):* Consider a channel with  $\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{s}_k + \mathbf{n}_k$ , where  $\mathbf{y}_k$  is the received vector,  $\mathbf{x}_k$  the transmitted vector,  $\mathbf{s}_k$  the vector Gaussian interference, and  $\mathbf{n}_k$  the vector white Gaussian noise. If  $\mathbf{s}_k$  and  $\mathbf{n}_k$  are independent and noncausal knowledge of  $\mathbf{s}_k$  is available at the transmitter but not at the receiver, then the capacity of the channel is the same as if  $\mathbf{s}_k$  were not present.

In the MIMO BC, this result can be applied at the transmitter when choosing codewords for different receivers. The transmitter first picks a codeword for receiver 1. The transmitter then chooses a codeword for receiver 2 with full (noncausal) knowledge of the codeword intended for receiver 1. Therefore, receiver 2 does not see the codeword intended for receiver 1 as interference. Similarly, the codeword for receiver 3 is chosen such that receiver 3 does not see the signals intended for receivers 1 and 2 as interference. This process continues for all  $K$  receivers. Receiver 1 subsequently sees the signals intended for all other users as interference, receiver 2 sees the signals intended for users 3 through  $K$  as interference, etc. Since the ordering of the users clearly matters in such a procedure, the following is an achievable rate vector:

$$R_{\pi(i)} = \log \frac{\left| \mathbf{I} + \mathbf{H}_{\pi(i)} \left( \sum_{j \geq i} \mathbf{\Sigma}_{\pi(j)} \right) \mathbf{H}_{\pi(i)}^\dagger \right|}{\left| \mathbf{I} + \mathbf{H}_{\pi(i)} \left( \sum_{j > i} \mathbf{\Sigma}_{\pi(j)} \right) \mathbf{H}_{\pi(i)}^\dagger \right|} \quad i = 1, \dots, K. \quad (3)$$

The dirty paper region  $\mathcal{C}_{\text{DPC}}(P, \mathbf{H})$  is defined as the convex hull of the union of all such rates vectors over all positive semidefinite covariance matrices  $\mathbf{\Sigma}_1, \dots, \mathbf{\Sigma}_K$  such that  $\text{Tr}(\mathbf{\Sigma}_1 + \dots + \mathbf{\Sigma}_K) = \text{Tr}(\mathbf{\Sigma}_x) \leq P$  and over all permutations  $(\pi(1), \dots, \pi(K))$

$$\mathcal{C}_{\text{DPC}}(P, \mathbf{H}) \triangleq \text{Co} \left( \bigcup_{\pi, \mathbf{\Sigma}_i} \mathbf{R}(\pi, \mathbf{\Sigma}_i) \right) \quad (4)$$

where  $\mathbf{R}(\pi, \mathbf{\Sigma}_i)$  is given by (3). The transmitted signal is  $\mathbf{x} = \mathbf{x}_1 + \dots + \mathbf{x}_K$  and the input covariance matrices are of the form  $\mathbf{\Sigma}_i = \mathbb{E}[\mathbf{x}_i \mathbf{x}_i^\dagger]$ . The dirty paper coding procedure yields statistically independent signals  $\mathbf{x}_1, \dots, \mathbf{x}_K$ , from which it follows that  $\mathbf{\Sigma}_x = \mathbf{\Sigma}_1 + \dots + \mathbf{\Sigma}_K$ .

One important feature to notice about the dirty paper rate equations in (3) is that the rate equations are neither a concave nor a convex function of the covariance matrices. This makes finding the dirty paper region very difficult, because generally the entire space of covariance matrices which meet the power constraint must be searched over. In this paper, we consider the dirty paper region subject to a transmit power constraint. Recent work [15] has characterized the dirty paper region subject to in-

dividual rate constraints (i.e., minimizing the transmit power required to achieve a certain set of rates).

### B. MIMO MAC Capacity Region

The capacity region of a general MIMO MAC was obtained in [2]–[4]. We now describe this capacity region for the dual MIMO MAC as defined in Section II. For any set of powers  $(P_1, \dots, P_K)$ , the capacity of the MIMO MAC is

$$\begin{aligned} \mathcal{C}_{\text{MAC}}(P_1, \dots, P_K; \mathbf{H}^\dagger) \\ \triangleq \bigcup_{\{\text{Tr}(\mathbf{P}_i) \leq P_i \forall i\}} \left\{ (R_1, \dots, R_K) : \sum_{i \in S} R_i \right. \\ \left. \leq \log \left| \mathbf{I} + \sum_{i \in S} \mathbf{H}_i^\dagger \mathbf{P}_i \mathbf{H}_i \right| \forall S \subseteq \{1, \dots, M\} \right\}. \end{aligned} \quad (5)$$

For  $P > 0$ , we denote by  $\mathcal{C}_{\text{union}}(P, \mathbf{H}^\dagger)$  the following set:

$$\begin{aligned} \mathcal{C}_{\text{union}}(P, \mathbf{H}^\dagger) \\ \triangleq \bigcup_{\sum_{i=0}^K P_i \leq P} \mathcal{C}_{\text{MAC}}(P_1, \dots, P_K; \mathbf{H}^\dagger) \\ = \bigcup_{\{\sum_{i=1}^K \text{Tr}(\mathbf{P}_i) \leq P\}} \left\{ (R_1, \dots, R_K) : \sum_{i \in S} R_i \right. \\ \left. \leq \log \left| \mathbf{I} + \sum_{i \in S} \mathbf{H}_i^\dagger \mathbf{P}_i \mathbf{H}_i \right| \forall S \subseteq \{1, \dots, M\} \right\}. \end{aligned} \quad (6)$$

By the argument provided in [4, Theorem 1], this region is convex. It can easily be shown that this region is the capacity region of a MAC when the transmitters have a sum power constraint instead of individual power constraints but are not allowed to cooperate. Additionally, the MIMO MAC rates can be shown to be a concave function of the covariance matrices. This implies that the boundary points (and the corresponding covariance matrices) of the sum power MIMO MAC capacity region can be found by a standard convex program (see [4] for a discussion of this with regards to the individual power constraint MIMO MAC, which is nearly identical in structure). This fact turns out to be quite important because later we show that the MIMO MAC sum power constraint region is equal to the dirty paper achievable region of the dual MIMO BC.

### C. Duality of the Scalar Gaussian MAC and BC

Finally, we state the duality result for scalar Gaussian MAC and BC channels [9].

*Theorem 1 (Jindal, Vishwanath, Goldsmith):* The capacity region of a scalar Gaussian BC with power  $P$  and channels  $\bar{h} = (h_1, \dots, h_K)$  is equal to the union of capacity regions of the dual MAC with powers  $(P_1, \dots, P_K)$  such that  $\sum_{i=1}^K P_i = P$

$$\mathcal{C}_{\text{BC}}(P; \bar{h}) = \bigcup_{\sum_{i=1}^K P_i = P} \mathcal{C}_{\text{MAC}}(P_1, \dots, P_K; \bar{h}).$$

The proof of this is obtained by showing that any set of rates achievable in the BC is also achievable in the MAC, and *vice*

*versa*. One key point is that to achieve the same rate vector in the BC and MAC, the decoding order must in general be reversed, i.e., if user 1 is decoded last in the BC then user 1 is decoded first in the MAC. In the next section, we will derive a similar result that equates the dirty paper BC achievable region with the union of MAC capacity regions for the MIMO channel we are considering.

#### IV. DUALITY OF THE MAC AND DIRTY PAPER BC REGION

In this section, we show that the capacity region of the MIMO MAC with a sum power constraint of  $P$  for the  $K$  transmitters is the same as the dirty paper region of the dual MIMO BC with power constraint  $P$ . In other words, any rate vector that is achievable in the dual MAC with power constraints  $(P_1, \dots, P_K)$  is in the dirty paper region of the BC with power constraint  $\sum_{i=1}^K P_i$ . Conversely, any rate vector that is in the dirty paper region of the BC is also in the dual MIMO MAC region with the same total power constraint.

*Theorem 2:* The dirty paper region of a MIMO BC channel with power constraint  $P$  is equal to the capacity region of the dual MIMO MAC with sum power constraint  $P$

$$\mathcal{C}_{\text{DPC}}(P, \mathbf{H}) = \mathcal{C}_{\text{union}}(P, \mathbf{H}^\dagger).$$

*Proof:* We first prove  $\mathcal{C}_{\text{DPC}}(P, \mathbf{H}) \supseteq \mathcal{C}_{\text{union}}(P, \mathbf{H}^\dagger)$  by showing that every rate vector achieved by successive decoding in the MAC is also in the dirty paper region of the dual MIMO BC. More specifically, we show by the MAC-to-BC transformations below that for every set of MAC covariance matrices  $\mathbf{P}_1, \dots, \mathbf{P}_K$  and any decoding order in the MAC, there exist BC covariance matrices  $\mathbf{\Sigma}_1, \dots, \mathbf{\Sigma}_K$  using the same sum power as the MAC (i.e.,  $\sum_{i=1}^K \text{Tr}(\mathbf{P}_i) = \sum_{i=1}^K \text{Tr}(\mathbf{\Sigma}_i)$ ) such that the MAC rates are achievable in the BC using the dirty paper coding method described in Section III-A. Each set of MAC covariance matrices corresponds to a  $K$ -dimensional polyhedron, as described in (5), with the  $K!$  corner points of the polyhedron corresponding to performing successive decoding at the receiver in one of the  $K!$  possible decoding orders. By the convexity of the dirty paper region (due to the convex hull operation), it is sufficient to show that the corner points of all polyhedrons (i.e., the successive decoding points) corresponding to all MAC covariance matrices are in the dirty paper region of the dual MIMO BC. Thus, with the MAC-to-BC transformations described below, this implies  $\mathcal{C}_{\text{DPC}}(P, \mathbf{H}) \supseteq \mathcal{C}_{\text{union}}(P, \mathbf{H}^\dagger)$ .

We complete the proof by showing

$$\mathcal{C}_{\text{DPC}}(P, \mathbf{H}) \subseteq \mathcal{C}_{\text{union}}(P, \mathbf{H}^\dagger).$$

We prove this by showing (via the BC-to-MAC transformations below) that for every set of BC covariance matrices and any encoding order there exist MAC covariance matrices that achieve the same set of rates using the same sum power. The convexity of the MIMO MAC sum power constraint region thus implies that

$$\mathcal{C}_{\text{DPC}}(P, \mathbf{H}) \subseteq \mathcal{C}_{\text{union}}(P, \mathbf{H}^\dagger).$$

This completes the proof, provided we have the transformations given below that map the MAC covariances to the BC covariances and *vice versa*.  $\square$

Next, we explain some terminology used in the transformations, followed by the actual transformations. It is important to point out that the transformations require a reverse decoding/encoding order of the users in the dual MAC/BC channel. In other words, if user 1 is decoded first in the MAC (i.e., user 1 suffers interference of all other users' signals), then we must encode user 1's signal *last* (i.e., no interference from other users) in the BC to achieve the same rates using these transformations. Also, notice that the proof of duality only requires *existence* of BC covariance matrices which satisfy the rates achieved by a set of MAC covariance matrices, and *vice versa*. However, the transformations that follow actually provide equations for the transformed BC covariances as a function of the MAC covariances, and *vice versa*. This can be quite useful because it is generally much easier to find the optimal BC covariance matrices by finding the optimal MIMO MAC covariance matrices and then transforming the matrices to BC covariance matrices (due to the convex structure of the MIMO MAC rate equations) than it is to directly search for the optimum BC covariance matrices.

#### A. Terminology

First, we explain the terms *effective channel* and *flipped channel*. A single-user MIMO system  $\Theta$  with channel matrix  $\mathbf{H}$ , additive Gaussian noise with covariance  $\mathbf{X}$ , and additive independent Gaussian interference with covariance  $\mathbf{Z}$  is said to have an effective channel of  $(\mathbf{X} + \mathbf{Z})^{-1/2}\mathbf{H}$ . The set of rates achievable by  $\Theta$  and a different system with channel matrix equal to the effective channel and with additive white noise of unit variance and no interference are the same. Also, the capacity of a system  $\Theta_1$  with effective channel matrix  $\mathbf{Y}$  and the capacity of system  $\Theta_2$  with effective channel matrix  $\mathbf{Y}^\dagger$ , termed the flipped channel, are the same [2]. In other words, for every transmit covariance  $\mathbf{\Sigma}$  in  $\Theta_1$ , there exists a  $\bar{\mathbf{\Sigma}}$  in  $\Theta_2$  with  $\text{Tr}(\bar{\mathbf{\Sigma}}) \leq \text{Tr}(\mathbf{\Sigma})$  such that the rate achieved by  $\bar{\mathbf{\Sigma}}$  in  $\Theta_2$  is equal to the rate achieved by  $\mathbf{\Sigma}$  in  $\Theta_1$ . In Appendix, Section A, we show that  $\bar{\mathbf{\Sigma}} = \mathbf{F}\mathbf{G}^\dagger\mathbf{\Sigma}\mathbf{G}\mathbf{F}^\dagger$  meets this criterion where the singular value decomposition (SVD) of  $\mathbf{Y}$  is  $\mathbf{Y} = \mathbf{F}\mathbf{\Lambda}\mathbf{G}^\dagger$  where  $\mathbf{\Lambda}$  is square and diagonal.<sup>1</sup> Next, we describe the covariance transformations.

#### B. MAC-to-BC Transformation

In this subsection, we derive a transformation that takes as inputs a set of MAC covariance matrices and a decoding order and outputs a set of BC covariances with the same sum power as the MAC covariances that achieve rates equal to the rates achieved in the MAC using the MAC covariance matrices and successive decoding with the specified decoding order.

Since the numbering of the users is arbitrary, we assume that user 1 is decoded first, user 2 second, and so on, at the MAC receiver.

Let

$$\mathbf{A}_j \triangleq \left( \mathbf{I} + \mathbf{H}_j \left( \sum_{l=1}^{j-1} \mathbf{\Sigma}_l \right) \mathbf{H}_j^\dagger \right)$$

<sup>1</sup>Note that the standard SVD command in MATLAB does not always return a square and diagonal matrix of singular values, so modification may be necessary to generate the flipped matrix correctly.

and

$$\mathbf{B}_j \triangleq \left( \mathbf{I} + \sum_{l=j+1}^K \mathbf{H}_l^\dagger \mathbf{P}_l \mathbf{H}_l \right).$$

The rate achieved by user  $j$  in the MAC for some arbitrary set of positive semidefinite covariance matrices  $(\mathbf{P}_1, \dots, \mathbf{P}_K)$  is given by

$$\begin{aligned} R_j^M &= \log \frac{\left| \mathbf{I} + \sum_{i=j}^K (\mathbf{H}_i^\dagger \mathbf{P}_i \mathbf{H}_i) \right|}{\left| \mathbf{I} + \sum_{i=j+1}^K (\mathbf{H}_i^\dagger \mathbf{P}_i \mathbf{H}_i) \right|}} \\ &= \log \left| \mathbf{I} + \left( \mathbf{I} + \sum_{i=j+1}^K (\mathbf{H}_i^\dagger \mathbf{P}_i \mathbf{H}_i) \right)^{-1} \mathbf{H}_j^\dagger \mathbf{P}_j \mathbf{H}_j \right| \\ &= \log \left| \mathbf{I} + \mathbf{B}_j^{-1} \mathbf{H}_j^\dagger \mathbf{P}_j \mathbf{H}_j \right|. \end{aligned}$$

Notice that  $\mathbf{B}_j$  represents the interference experienced by user  $j$  in the MAC. To simplify, we take the square root of  $\mathbf{B}_j^{-1}$  and use the property  $|\mathbf{I} + \mathbf{AB}| = |\mathbf{I} + \mathbf{BA}|$ . We also introduce  $\mathbf{A}_j^{-1/2} \mathbf{A}_j^{1/2} = \mathbf{I}$  into the expression to get

$$R_j^M = \log \left| \mathbf{I} + \mathbf{B}_j^{-1/2} \mathbf{H}_j^\dagger \mathbf{A}_j^{-1/2} \mathbf{A}_j^{1/2} \mathbf{P}_j \mathbf{A}_j^{1/2} \mathbf{A}_j^{-1/2} \mathbf{H}_j \mathbf{B}_j^{-1/2} \right|.$$

Treating  $\mathbf{B}_j^{-1/2} \mathbf{H}_j^\dagger \mathbf{A}_j^{-1/2}$  as the effective channel of the system, we flip the channel and find  $\overline{\mathbf{A}_j^{1/2} \mathbf{P}_j \mathbf{A}_j^{1/2}}$  such that

$$\begin{aligned} \overline{\text{Tr}(\mathbf{A}_j^{1/2} \mathbf{P}_j \mathbf{A}_j^{1/2})} &\leq \text{Tr}(\mathbf{A}_j^{1/2} \mathbf{P}_j \mathbf{A}_j^{1/2}) \\ R_j^M &= \log \left| \mathbf{I} + \mathbf{A}_j^{-1/2} \mathbf{H}_j \mathbf{B}_j^{-1/2} \overline{\mathbf{A}_j^{1/2} \mathbf{P}_j \mathbf{A}_j^{1/2}} \mathbf{B}_j^{-1/2} \mathbf{H}_j^\dagger \mathbf{A}_j^{-1/2} \right|. \end{aligned}$$

Now consider the rate of user  $j$  in the BC assuming that the opposite encoding order is used (i.e., user 1 is encoded last, user 2 second to last, etc.)

$$\begin{aligned} R_j^B &= \log \frac{\left| \mathbf{I} + \sum_{i=1}^j (\mathbf{H}_j \mathbf{\Sigma}_i \mathbf{H}_j^\dagger) \right|}{\left| \mathbf{I} + \sum_{i=1}^{j-1} (\mathbf{H}_j \mathbf{\Sigma}_i \mathbf{H}_j^\dagger) \right|}} \\ &= \log \left| \mathbf{I} + \mathbf{A}_j^{-1} \mathbf{H}_j \mathbf{\Sigma}_j \mathbf{H}_j^\dagger \right| \\ &= \log \left| \mathbf{I} + \mathbf{A}_j^{-1/2} \mathbf{H}_j \mathbf{\Sigma}_j \mathbf{H}_j^\dagger \mathbf{A}_j^{-1/2} \right|. \end{aligned}$$

Here  $\mathbf{A}_j$  represents the interference experienced by user  $j$  in the BC. If we chose the BC covariances as

$$\mathbf{\Sigma}_1 = \mathbf{B}_1^{-1/2} \overline{\mathbf{P}_1} \mathbf{B}_1^{-1/2} \quad (8)$$

⋮

$$\mathbf{\Sigma}_j = \mathbf{B}_j^{-1/2} \overline{\mathbf{A}_j^{1/2} \mathbf{P}_j \mathbf{A}_j^{1/2}} \mathbf{B}_j^{-1/2} \quad (9)$$

⋮

$$\mathbf{\Sigma}_K = \overline{\mathbf{A}_K^{1/2} \mathbf{P}_K \mathbf{A}_K^{1/2}} \quad (10)$$

clearly we see  $R_j^M = R_j^B$ . Additionally, it is easy to show that the resulting covariance matrices are all symmetric and positive semidefinite. In Appendix, Section B, we show that the transformations given by (8)–(10) satisfy the sum trace constraint, or that  $\sum_{i=1}^K \text{Tr}(\mathbf{\Sigma}_i) \leq \sum_{i=1}^K \text{Tr}(\mathbf{P}_i)$ . Note that  $\mathbf{\Sigma}_j$  depends

only on  $\mathbf{\Sigma}_1, \dots, \mathbf{\Sigma}_{j-1}$ , and hence, the  $\mathbf{\Sigma}_j$  can be computed sequentially in increasing order. By doing this for all  $K$  users, we find covariance matrices for the BC that achieve the same rate as in the MAC. If we substitute in the expression generating the flipped channel, the expression for the BC covariance matrix of the  $j$ th user (9) can be expanded as

$$\mathbf{\Sigma}_j = \mathbf{B}_j^{-1/2} \mathbf{F}_j \mathbf{G}_j^\dagger \mathbf{A}_j^{1/2} \mathbf{P}_j \mathbf{A}_j^{1/2} \mathbf{G}_j \mathbf{F}_j^\dagger \mathbf{B}_j^{-1/2} \quad (11)$$

where the effective channel  $\mathbf{B}_j^{-1/2} \mathbf{H}_j^\dagger \mathbf{A}_j^{-1/2}$  is decomposed using the SVD as  $\mathbf{B}_j^{-1/2} \mathbf{H}_j^\dagger \mathbf{A}_j^{-1/2} = \mathbf{F}_j \mathbf{\Lambda}_j \mathbf{G}_j^\dagger$ , where  $\mathbf{\Lambda}_j$  is a square and diagonal matrix.

### C. BC-to-MAC Transformation

In this subsection, we derive a transformation which, given a set of BC covariance matrices and an encoding order, outputs a set of MAC covariances with the same sum power as the BC covariances that achieve MAC rates (using successive decoding) equal to the rates achieved in the BC using the BC covariance matrices. These transformations are almost identical to the MAC-to-BC transformations. For the dirty paper encoding at the BC, we assume that user  $K$  is encoded first, user  $K-1$  second, and so on, in decreasing order. Along the same lines as the MAC–BC transformation, we treat  $\mathbf{A}_j^{-1/2} \mathbf{H}_j \mathbf{B}_j^{-1/2}$  as the effective channel and  $\mathbf{B}_j^{1/2} \mathbf{\Sigma}_j \mathbf{B}_j^{1/2}$  as the covariance matrix. By flipping the effective channel, we obtain  $\overline{\mathbf{B}_j^{1/2} \mathbf{\Sigma}_j \mathbf{B}_j^{1/2}}$  and obtain the transformation

$$\mathbf{P}_K = \mathbf{A}_K^{-1/2} \overline{\mathbf{\Sigma}_K} \mathbf{A}_K^{-1/2} \quad (12)$$

⋮

$$\mathbf{P}_j = \mathbf{A}_j^{-1/2} \overline{\mathbf{B}_j^{1/2} \mathbf{\Sigma}_j \mathbf{B}_j^{1/2}} \mathbf{A}_j^{-1/2} \quad (13)$$

⋮

$$\mathbf{P}_1 = \overline{\mathbf{B}_1^{1/2} \mathbf{\Sigma}_1 \mathbf{B}_1^{1/2}}. \quad (14)$$

As before, if we use the opposite decoding order in the MAC (i.e., user 1 decoded first, etc.), this transformation ensures that the rates of all users in the BC and MAC are equivalent along with the total power used in the BC and MAC. Also note that we can sequentially compute the  $\mathbf{P}_j$ 's in decreasing numerical order. If we substitute in the expression generating the flipped channel, the expression for the MAC covariance matrix of the  $j$ th user in (13) can be expanded as

$$\mathbf{P}_j = \mathbf{A}_j^{-1/2} \mathbf{F}_j \mathbf{G}_j^\dagger \mathbf{B}_j^{1/2} \overline{\mathbf{\Sigma}_j} \mathbf{B}_j^{1/2} \mathbf{G}_j \mathbf{F}_j^\dagger \mathbf{A}_j^{-1/2} \quad (15)$$

where the effective channel  $\mathbf{A}_j^{-1/2} \mathbf{H}_j \mathbf{B}_j^{-1/2}$  is decomposed using the SVD as  $\mathbf{A}_j^{-1/2} \mathbf{H}_j \mathbf{B}_j^{-1/2} = \mathbf{F}_j \mathbf{\Lambda}_j \mathbf{G}_j^\dagger$ , where  $\mathbf{\Lambda}_j$  is a square and diagonal matrix.

### D. MIMO MAC with Individual Power Constraints

We can also obtain the capacity region of a MIMO MAC with individual power constraints on each user from the dirty paper region of a dual MIMO BC. By [9, Theorem 3], we can characterize the individual power constraint MIMO MAC capacity region as an intersection of sum power constraint MIMO MAC capacity regions. By duality, we know that the sum power constraint MIMO MAC capacity region is equal to the dirty paper achievable region of the dual MIMO BC.

*Corollary 1:* The capacity region of a MIMO MAC is the intersection of the scaled dirty paper regions of the MIMO BC. Mathematically, this is stated as

$$\begin{aligned} \mathcal{C}_{\text{MAC}}(P_1, \dots, P_K, \mathbf{H}^\dagger) \\ = \bigcap_{\alpha > 0} \mathcal{C}_{\text{DPC}} \left( \sum_{i=1}^K \frac{P_i}{\alpha_i}; [\sqrt{\alpha_1} \mathbf{H}_1^T \dots \sqrt{\alpha_K} \mathbf{H}_K^T]^T \right). \end{aligned} \quad (16)$$

*Proof:* This result can be obtained by a straightforward application of [9, Theorem 3] to the MIMO MAC capacity region and the duality developed in Theorem 2. The scaled MIMO BC here refers to the channel where the matrix of each receiver  $\mathbf{H}_i$  is scaled by  $\sqrt{\alpha_i}$ .  $\square$

## V. SUM RATE CAPACITY OF BC CHANNELS

In the previous sections we showed that the dirty paper region of the BC and the union of the dual MAC capacity regions are equivalent. Now we show that the dirty paper broadcasting strategy is the capacity-achieving strategy for the sum rate capacity of the MIMO BC. To do this, we make use of the duality of the dirty paper region and the dual MAC to show that the dirty paper region achieves an upper bound on the sum rate capacity of the MIMO BC.

In [10], Sato presents an upper bound on the capacity region of general BCs. This bound utilizes the capacity of the *cooperative system* as defined in Section II. Since the cooperative system is the same as the BC, but with receiver coordination, the capacity of the cooperative system ( $\mathcal{C}_{\text{coop}}(P, \mathbf{H})$ ) is an upper bound on the BC sum rate capacity ( $\mathcal{C}_{\text{BC}}^{\text{sum rate}}(P, \mathbf{H})$ ). This bound is not tight in general, but by introducing noise correlation at the different receivers, we can get a much stronger bound.

Since the capacity region of a general BC depends only on the marginal transition probabilities of the channel (i.e.,  $p(y_i|x)$ ) and not on the joint distribution  $p(y_1, \dots, y_K|x)$  [16, Theorem 14.6.1], correlation between the noise vectors at *different receivers* of the BC does not affect the BC capacity region. It does, however, affect the capacity of the cooperative system, which is still an upper bound on the sum rate of the BC. Therefore, we retain  $\mathbb{E}(\mathbf{n}_i \mathbf{n}_i^\dagger) = \mathbf{I}$ ,  $1 \leq i \leq K$  as before (i.e., noise components at the multiple receive antennas within a single receiver are uncorrelated) and let  $\mathbb{E}(\mathbf{n}_i \mathbf{n}_j^\dagger) \triangleq \mathbf{X}_{i,j} \prec \mathbf{I}$ , since introducing noise correlation within a receiver affects the broadcast capacity region. Let  $\mathbf{Z}$  denote the noise covariance matrix in the cooperative system (i.e.  $\mathbf{Z} = E[\mathbf{z}\mathbf{z}^T]$  where  $\mathbf{z} = [\mathbf{n}_1 \dots \mathbf{n}_K]^T$ ) to define the set  $S$  to be all *nonsingular* (or strictly positive definite) noise covariance matrices satisfying the Sato upper bound conditions

$$S = \left\{ \mathbf{Z} : \mathbf{Z} > 0, \mathbf{Z} = \begin{bmatrix} \mathbf{I} & \dots & \mathbf{X}_{K,1}^\dagger \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{K,1} & \dots & \mathbf{I} \end{bmatrix} \right\}. \quad (17)$$

Then, for any  $\mathbf{Z} \in S$ , the cooperative capacity  $\mathcal{C}_{\text{coop}}(P, \mathbf{Z}^{-1/2} \mathbf{H})$  is an upper bound to  $\mathcal{C}_{\text{BC}}^{\text{sum rate}}(P, \mathbf{H})$ , since receiver coordination can only increase capacity. Hence, an upper bound on  $\mathcal{C}_{\text{BC}}^{\text{sum rate}}(P, \mathbf{H})$  can be obtained as

$$\mathcal{C}_{\text{BC}}^{\text{sum rate}}(P, \mathbf{H}) \leq \inf_{\mathbf{Z} \in S} \mathcal{C}_{\text{coop}}(P, \mathbf{Z}^{-1/2} \mathbf{H}).$$

From [2], the cooperative capacity is defined as

$$\begin{aligned} \mathcal{C}_{\text{coop}}(P, \mathbf{Z}^{-1/2} \mathbf{H}) \\ = \max_{\mathbf{\Sigma} \geq 0, \text{Tr}(\mathbf{\Sigma}) \leq P} \log |\mathbf{I} + \mathbf{Z}^{-1/2} \mathbf{H} \mathbf{\Sigma} \mathbf{H}^\dagger \mathbf{Z}^{-1/2}|. \end{aligned}$$

Using this definition, we write the Sato upper bound as

$$\begin{aligned} \mathcal{C}_{\text{Sato}}(P, \mathbf{H}) \\ \triangleq \inf_{\mathbf{Z} \in S} \max_{\mathbf{\Sigma} \geq 0, \text{Tr}(\mathbf{\Sigma}) \leq P} \log |\mathbf{I} + \mathbf{Z}^{-1/2} \mathbf{H} \mathbf{\Sigma} \mathbf{H}^\dagger \mathbf{Z}^{-1/2}|. \end{aligned} \quad (18)$$

Thus, the region given by

$$\left\{ (R_1, \dots, R_K) : \sum_{i=1}^K R_i \leq \mathcal{C}_{\text{Sato}}(P, \mathbf{H}) \right\}$$

is an upper bound on  $\mathcal{C}_{\text{BC}}(P, \mathbf{H})$ . Next, we show that the sum rate capacity of the MIMO BC actually equals the Sato upper bound.

*Theorem 3:* The sum rate capacity of the MIMO BC equals the Sato upper bound. Furthermore, the dirty paper coding strategy achieves the sum rate capacity of the MIMO BC

$$\mathcal{C}_{\text{BC}}^{\text{sum rate}}(P, \mathbf{H}) = \mathcal{C}_{\text{DPC}}^{\text{sum rate}}(P, \mathbf{H}) = \mathcal{C}_{\text{Sato}}(P, \mathbf{H}). \quad (19)$$

*Proof:* Since the sum rate capacity of the MIMO BC can be no larger than the Sato upper bound, it is sufficient to show that the Sato upper bound is actually achievable in the MIMO BC using dirty paper coding. Note that, by duality, we know that the maximum sum rate of the MIMO MAC equals the maximum sum rate of the dirty paper achievable region. We, therefore, must show the inequality

$$\begin{aligned} \mathcal{C}_{\text{DPC}}^{\text{sum rate}}(P, \mathbf{H}) &= \mathcal{C}_{\text{union}}^{\text{sum rate}}(P, \mathbf{H}) \\ &= \max_{\{\mathbf{P}_j \geq 0, \sum_{i=1}^K \text{Tr}(\mathbf{P}_i) \leq P\}} \\ &\quad \cdot \log \left| \mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger \mathbf{P}_i \mathbf{H}_i \right| \\ &\geq \mathcal{C}_{\text{Sato}}(P, \mathbf{H}). \end{aligned} \quad (20)$$

We prove (21) by using Lagrangian duality to express both the Sato upper bound and the MIMO MAC sum rate capacity in different forms. Specifically, as shown in Appendix, Section C, we can alternatively write the Sato upper bound defined in (18) as (see (58))

$$\begin{aligned} \mathcal{C}_{\text{Sato}}(P, \mathbf{H}) &= \inf_{\mathbf{Z} \in S} \min_{\mathbf{A}, \lambda} -\log |\mathbf{A}| + \text{Tr}(\mathbf{A}) + \lambda P - t \\ &\quad \text{such that} \\ &\quad \lambda \geq 0, \mathbf{A} \geq 0 \\ &\quad \lambda \mathbf{Z} \geq \mathbf{H} \mathbf{A} \mathbf{H}^\dagger \end{aligned} \quad (22)$$

and the MIMO MAC sum rate capacity (20) as

$$\begin{aligned} \mathcal{C}_{\text{union}}^{\text{sum rate}}(P, \mathbf{H}) &= \min_{\mathbf{A}, \lambda} -\log |\mathbf{A}| + \text{Tr}(\mathbf{A}) + \lambda P - t \\ &\quad \text{such that} \\ &\quad \lambda \geq 0, \mathbf{A} \geq 0, \\ &\quad \lambda \mathbf{I} \geq \mathbf{H}_i \mathbf{A} \mathbf{H}_i^\dagger \forall i. \end{aligned} \quad (23)$$

Notice that the objective functions of (22) and (23) are the same, but the variables and constraints are different. We will show that  $\mathcal{C}_{\text{union}}^{\text{sum rate}}(P, \mathbf{H}) \geq \mathcal{C}_{\text{Sato}}(P, \mathbf{H})$  by constructing a feasible

solution to the Sato upper bound dual problem from an optimal solution to the MIMO MAC sum rate dual problem.

Let  $\lambda = \lambda_0$ ,  $\mathbf{A} = \mathbf{A}_0$  be an optimizing solution to (23), i.e.,

$$C_{\text{union}}^{\text{sum rate}}(P, \mathbf{H}) = -\log |\mathbf{A}_0| + \text{Tr}(\mathbf{A}_0) + \lambda_0 P - t.$$

Since (23) is a minimization of a convex function over a closed set, we know that the minimum is achieved, so a minimizing pair  $(\lambda_0, \mathbf{A}_0)$  exists. We prove (21) by explicitly constructing a feasible set of variables  $(\lambda, \mathbf{A}, \mathbf{Z})$  for the Sato upper bound (22) such that the objective functions in (23) and (22) are equal.

Let us first consider the choice of values of  $(\lambda, \mathbf{A}, \mathbf{Z})$  as

$$\lambda = \lambda_0 \quad (24)$$

$$\mathbf{A} = \mathbf{A}_0 \quad (25)$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I} & \frac{\mathbf{H}_1 \mathbf{A}_0 \mathbf{H}_2^\dagger}{\lambda_0} & \cdots & \frac{\mathbf{H}_1 \mathbf{A}_0 \mathbf{H}_K^\dagger}{\lambda_0} \\ \frac{\mathbf{H}_2 \mathbf{A}_0 \mathbf{H}_1^\dagger}{\lambda_0} & \mathbf{I} & \cdots & \frac{\mathbf{H}_2 \mathbf{A}_0 \mathbf{H}_K^\dagger}{\lambda_0} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathbf{H}_K \mathbf{A}_0 \mathbf{H}_1^\dagger}{\lambda_0} & \frac{\mathbf{H}_K \mathbf{A}_0 \mathbf{H}_2^\dagger}{\lambda_0} & \cdots & \mathbf{I} \end{bmatrix}. \quad (26)$$

As long as this choice of  $\mathbf{Z}$  is positive definite, it can be verified by the method used below that this set is feasible for (22) and that the objective functions of both minimizations are the same. Thus, we have constructed the worst case noise  $\mathbf{Z}$  for the Sato upper bound and have shown (21), but only for the case when  $\mathbf{Z} > 0$ . However, in many practical cases, this choice of  $\mathbf{Z}$  is singular, and hence, not a feasible choice of  $\mathbf{Z}$  for (22).

To circumvent this singularity, we construct, instead, a family of feasible points (i.e.,  $\mathbf{Z} > 0$ ) by introducing an arbitrary parameter  $\delta > 0$ . This family of values of  $(\lambda, \mathbf{A}, \mathbf{Z})$  is given by

$$\lambda = \lambda_0 + \delta \quad (27)$$

$$\mathbf{A} = \mathbf{A}_0 \quad (28)$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I} & \frac{\mathbf{H}_1 \mathbf{A}_0 \mathbf{H}_2^\dagger}{\lambda_0 + \delta} & \cdots & \frac{\mathbf{H}_1 \mathbf{A}_0 \mathbf{H}_K^\dagger}{\lambda_0 + \delta} \\ \frac{\mathbf{H}_2 \mathbf{A}_0 \mathbf{H}_1^\dagger}{\lambda_0 + \delta} & \mathbf{I} & \cdots & \frac{\mathbf{H}_2 \mathbf{A}_0 \mathbf{H}_K^\dagger}{\lambda_0 + \delta} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathbf{H}_K \mathbf{A}_0 \mathbf{H}_1^\dagger}{\lambda_0 + \delta} & \frac{\mathbf{H}_K \mathbf{A}_0 \mathbf{H}_2^\dagger}{\lambda_0 + \delta} & \cdots & \mathbf{I} \end{bmatrix}. \quad (29)$$

We need to ensure that this set is feasible for (22). Since  $(\lambda_0, \mathbf{A}_0)$  are an optimizing solution of (23),  $(\lambda_0, \mathbf{A}_0)$  must satisfy the constraints in (23). Therefore, we have that  $\lambda = \lambda_0 + \delta > 0$  and  $\mathbf{A} = \mathbf{A}_0 \geq 0$ . Since the matrix  $\mathbf{Z}$  is block diagonal and symmetric by construction, we see that if  $\mathbf{Z} > 0$  then  $\mathbf{Z} \in \mathcal{S}$ . We, thus, need to verify that  $\mathbf{Z} > 0$  and that  $\lambda \mathbf{Z} \geq \mathbf{H} \mathbf{A} \mathbf{H}^\dagger$ .

Note that  $\lambda_0 + \delta > 0$  and

$$\begin{aligned} (\lambda_0 + \delta) \mathbf{Z} - \mathbf{H} \mathbf{A}_0 \mathbf{H}^\dagger &= \text{diag}[(\lambda_0 + \delta) \mathbf{I} - \mathbf{H}_i \mathbf{A}_0 \mathbf{H}_i^\dagger] \\ &= \text{diag}[\lambda_0 \mathbf{I} - \mathbf{H}_i \mathbf{A}_0 \mathbf{H}_i^\dagger] + \delta \mathbf{I}. \end{aligned} \quad (30)$$

Since  $(\lambda_0, \mathbf{A}_0)$  are an optimizing solution of (23), we have  $\lambda_0 \mathbf{I} - \mathbf{H}_i \mathbf{A}_0 \mathbf{H}_i^\dagger \geq 0$  for all  $i$ . This implies that  $\text{diag}[\lambda_0 \mathbf{I} - \mathbf{H}_i \mathbf{A}_0 \mathbf{H}_i^\dagger] \geq 0$ . Since  $\delta > 0$ , we have

$$\lambda \mathbf{Z} - \mathbf{H} \mathbf{A}_0 \mathbf{H}^\dagger = \text{diag}[\lambda_0 \mathbf{I} - \mathbf{H}_i \mathbf{A}_0 \mathbf{H}_i^\dagger] + \delta \mathbf{I} > 0. \quad (31)$$

This implies that  $\lambda \mathbf{Z} > \mathbf{H} \mathbf{A}_0 \mathbf{H}^\dagger = \mathbf{H} \mathbf{A} \mathbf{H}^\dagger$ . It, thus, remains to show that  $\mathbf{Z} > 0$ . Since  $\mathbf{A} \geq 0$ , we also have that  $\mathbf{H} \mathbf{A} \mathbf{H}^\dagger \geq 0$ . This implies that  $\lambda \mathbf{Z} > 0$ . Since  $\lambda > 0$ , we then get  $\mathbf{Z} > 0$ . Hence,  $(\lambda, \mathbf{A}, \mathbf{Z})$  form a feasible set of values for (22). Since

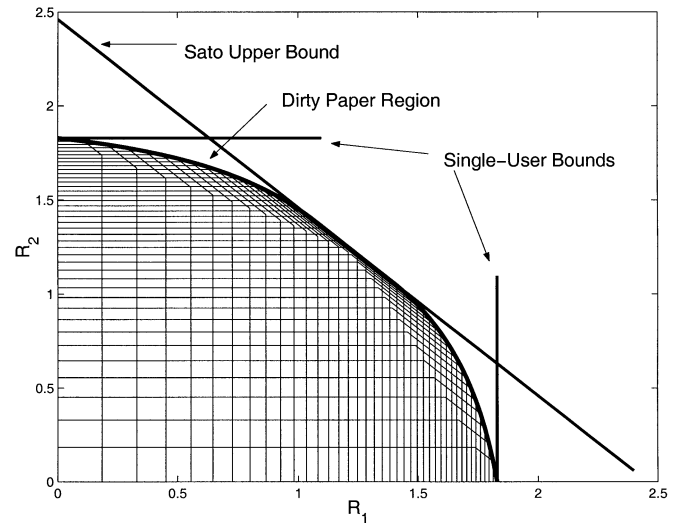


Fig. 2. Dirty paper broadcast region:  $K = 2, t = 2, r_1 = r_2 = 1, H_1 = [1 \ 4], H_2 = [4 \ 1], P = 10$ .

the Sato upper bound is equal to the infimum (over the feasible set) of the objective function, we have

$$\begin{aligned} C_{\text{Sato}}(P, \mathbf{H}) &\leq -\log |\mathbf{A}| + \text{Tr}(\mathbf{A}) + \lambda P - t \\ &= -\log |\mathbf{A}_0| + \text{Tr}(\mathbf{A}_0) + \lambda_0 P - t + \delta P \\ &= C_{\text{union}}^{\text{sum rate}}(P, \mathbf{H}) + \delta P. \end{aligned} \quad (32)$$

Since this holds for any  $\delta > 0$ , we get

$$C_{\text{Sato}}(P, \mathbf{H}) \leq C_{\text{union}}^{\text{sum rate}}(P, \mathbf{H})$$

which completes the proof of the theorem.  $\square$

A nice property of the proof of the sum rate capacity is that it is *constructive* in the sense that the proof generates worst case noise covariances for the Sato upper bound, assuming that the optimizing solution to the MIMO MAC sum rate problem is known. Specifically, in (29), we explicitly construct a noise covariance for which the cooperative capacity is larger than the MIMO MAC sum rate capacity by an arbitrarily small amount  $\delta P$ . Though we show that the constructed matrix  $\mathbf{Z} > 0$  when  $\delta > 0$ , as noted earlier when  $\delta \geq 0$  we still are guaranteed  $\mathbf{Z} \geq 0$ . If the constructed matrix  $\mathbf{Z}$  with  $\delta = 0$  is strictly non-singular (i.e.,  $\mathbf{Z} > 0$ ), then the cooperative capacity with noise covariance  $\mathbf{Z}$  is *equal* to the MIMO MAC sum rate capacity. Therefore, in these cases,  $\mathbf{Z}$  is in fact a worst case noise covariance for the Sato upper bound. Numerically, we also find  $\mathbf{Z}$  to be a worst case noise covariance for cases when  $\mathbf{Z}$  is singular when  $\delta = 0$ , as shown later in Example 2.

The dirty paper BC region and the capacity regions of the dual MAC, along with the Sato upper bound and single-user bounds, are illustrated for a symmetric two user channel in Fig. 2. The dirty paper region is the union of the pentagons in the figure because the dirty paper region is formed as a union of the individual power constraint MAC regions. Since each receiver has only a single antenna, the dual MIMO MAC region with individual power constraints is a simple pentagon. The capacity upper bound is obtained by taking the intersection of the regions formed by the two single-user optimum corner points (which are parallel to the axes) and the Sato upper bound, which is tight at the sum rate capacity. Note that the region formed by all three

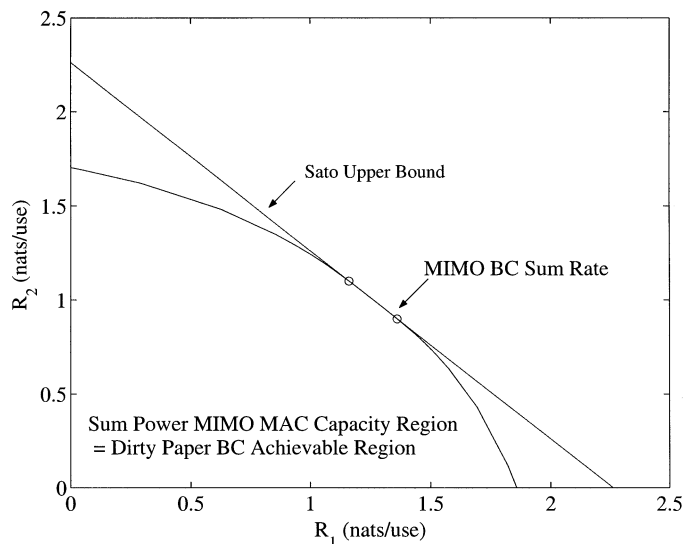


Fig. 3. Achievable region and Sato upper bound for Example 1

upper bounds is, in fact, quite close to the dirty paper achievable region. Also note that the boundary of the dirty paper achievable region has a straight-line segment at the sum rate point. This characteristic of possessing a straight-line segment (i.e., a time-division portion) at sum rate is also true for the MIMO MAC capacity region when the transmitters have more than one antenna [4].

## VI. NUMERICAL EXAMPLES

In this section, we provide two numerical examples to better illustrate the concepts discussed in the paper.

*Example 1:* Consider a two-user BC with  $P = 1$  and channel matrices

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0.8 \\ 0.5 & 2 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} 0.2 & 1 \\ 2 & 0.5 \end{bmatrix}. \quad (33)$$

The dirty paper achievable region is very difficult to compute without employing duality, as discussed in Section III-A. Thus, we find the dual MAC region using convex optimization techniques to obtain the achievable region in Fig. 3.

To compute the Sato upper bound for the problem, we solve the dual problem to the MIMO MAC. Note that this problem is a convex problem with linear matrix inequality constraints. There are many techniques in convex optimization literature to solve such problems. We use an easily available software called SDPSOL, developed by Boyd and Wu [17], to get that  $C_{\text{union}}^{\text{sum rate}}(P, \mathbf{H}) = 2.2615$  nats/s and obtain  $\mathbf{A}_0$ ,  $\lambda_0$  to be

$$\mathbf{A}_0 = \begin{bmatrix} 3.231 & 1.43 \\ 1.43 & 3.603 \end{bmatrix} \\ \lambda_0 = 1.2879.$$

From these, we obtain the worst case noise (using (29) with  $\delta = 0$ )

$$\mathbf{Z} = \begin{bmatrix} 1 & 0 & 0.1332 & 0.4446 \\ 0 & 1 & 0.4478 & 0.0613 \\ 0.1332 & 0.4478 & 1 & 0 \\ 0.4446 & 0.0613 & 0 & 1 \end{bmatrix}$$

which is nonsingular. Therefore, we find that

$$C_{\text{coop}}(P, \mathbf{Z}^{-1/2} \mathbf{H}) = C_{\text{Sato}}(P, \mathbf{H}) = C_{\text{union}}^{\text{sum rate}}(P, \mathbf{H}).$$

The corresponding upper bound to the capacity region is shown in Fig. 3.

The sum rate maximizing covariance matrices in the MAC are

$$\mathbf{P}_1 = \begin{bmatrix} 0.0720 & 0.1827 \\ 0.1827 & 0.4634 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 0 & 0.0026 \\ 0.0026 & 0.4646 \end{bmatrix}.$$

Notice that the sum rate is not maximized at a single point on the boundary of the capacity region, but it is actually maximized along a line segment. The corner points of this line segment are circled in Fig. 3. In the MAC, the lower corner point of this line segment can be achieved using the above covariance matrices  $\mathbf{P}_1$ ,  $\mathbf{P}_2$  and by decoding user 1 last. The upper corner point of the line segment can be achieved using the *same* covariance matrices, but the *opposite* decoding order (i.e., decode user 2 last). Any other point on the line segment can be achieved by time-sharing between these two decoding orders.

We can use the MAC-BC transformations in (8) to find the corresponding sum rate capacity-achieving BC covariance matrices. Note, however, that the transformations depend on the decoding order in the MAC. If we assume that user 1 is decoded last in the MAC, the transformed BC covariances are

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 0.0746 & 0.1932 \\ 0.1932 & 0.5004 \end{bmatrix}, \quad \mathbf{\Sigma}_2 = \begin{bmatrix} 0.4104 & -0.0776 \\ -0.0776 & 0.0147 \end{bmatrix}.$$

The lower corner point of the sum rate line segment is then achievable in the BC using these covariance matrices and by decoding user 2 last (i.e using dirty paper coding for user 2 to cancel out the signal of user 1). To achieve the upper corner point of the sum rate line segment, we must perform the MAC-BC transformations using the opposite order in the MAC. We then get

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 0.0001 & -0.0069 \\ -0.0069 & 0.4841 \end{bmatrix}, \quad \mathbf{\Sigma}_2 = \begin{bmatrix} 0.4849 & 0.1225 \\ 0.1225 & 0.0309 \end{bmatrix}.$$

Clearly, these BC covariance matrices are different than those used to achieve the other corner point of the sum rate line segment. Therefore, we see that in the MAC the corner points of the sum rate boundary can be achieved by using the same set of covariance matrices and different decoding orders. In the BC, however, a different decoding order and different covariance matrices are needed to achieve the corner points of the sum rate boundary.

*Example 2:* Consider a three-user BC, with two antennas at the transmitter ( $t = 2$ ) and one antenna each per receiver ( $r_1 = r_2 = r_3 = 1$ ). The channel matrices are given by  $\mathbf{H}_1 = [0 \ 1]$ ,  $\mathbf{H}_2 = [-\sqrt{3}/2 \ -1/2]$ ,  $\mathbf{H}_3 = [\sqrt{3}/2 \ -1/2]$ , and the total power constraint is  $P = 1$ . Note that the channels are unit vectors in Euclidean space, and are spaced  $120^\circ$  apart, as shown in Fig. 4. Also note that the channel matrix  $\mathbf{H}$

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -\sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \quad (34)$$

has rank two, and that  $\mathbf{H}_1 = -(\mathbf{H}_2 + \mathbf{H}_3)$ .



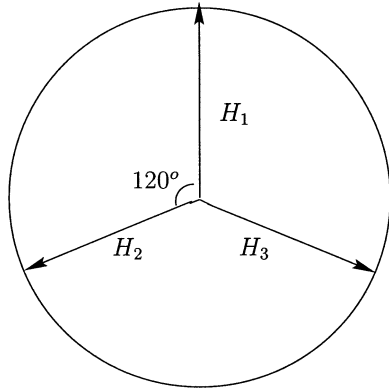


Fig. 4. Channel parameters for Example 2.

First, let us consider the dual MAC problem. By the symmetric structure of these channels, it is clear that allocating equal power to each user maximizes the sum rate of this system. Thus sum rate capacity is achieved with  $P_1 = P_2 = P_3 = 1/3$  and any MAC decoding order.

Using the MAC-to-BC transformations in (8), we find covariances in the broadcast (corresponding to encoding user 1 last, user 2 second, and user 3 first) that achieve the same sum rate point on the capacity region to be

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.2857 \end{bmatrix}, \quad \mathbf{\Sigma}_2 = \begin{bmatrix} 0.2187 & 0.1623 \\ 0.1623 & 0.1205 \end{bmatrix} \quad (35)$$

$$\mathbf{\Sigma}_3 = \begin{bmatrix} 0.2812 & -0.1624 \\ -0.1624 & 0.0937 \end{bmatrix} \quad (36)$$

and the sum rate capacity equals

$$\log \left| \mathbf{I} + \frac{1}{3} (\mathbf{H}_1^\dagger \mathbf{H}_1 + \mathbf{H}_2^\dagger \mathbf{H}_2 + \mathbf{H}_3^\dagger \mathbf{H}_3) \right| = 0.8109 \text{ nats/use.}$$

Now, let us find the worst case noise for the Sato upper bound. Note that, if  $n_1 = -(n_2 + n_3)$ , then  $y_1 = -(y_2 + y_3)$ . This implies that the received signal at one of the antennas is a linear combination of the signal at the other two antennas. Therefore, one receive antenna can be eliminated from the system without any loss in the cooperative capacity. Since  $\mathbb{E}(n_1^2) = \mathbb{E}(n_2^2) = \mathbb{E}(n_3^2) = 1$ , we require that  $\mathbb{E}(n_i n_j) = -0.5 \forall i \neq j$ . Thus, a noise covariance matrix given by

$$\mathbf{Q}_{\text{cand}} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix} \quad (37)$$

corresponds to  $n_1 = -(n_2 + n_3)$ , which allows us to eliminate one receive antenna. This noise covariance is a candidate for the worst case noise covariance for the Sato upper bound. Note that this noise covariance is singular. If we eliminate the third antenna output, we are left with the following two-input-two-output channel:

$$\mathbf{G} = \begin{bmatrix} 0 & 1 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}. \quad (38)$$

The cooperative capacity with this noise covariance then is

$$\max_{\mathbf{\Sigma}} \log \left| \mathbf{I} + \begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}^{-1} \mathbf{G} \mathbf{\Sigma} \mathbf{G}^\dagger \right|. \quad (39)$$

To evaluate the expression above, we use the standard water-filling technique [2] and find the optimizing  $\mathbf{\Sigma}$  to be

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

The corresponding cooperative capacity is thus equal to 0.8109 nats/channel use. Since the cooperative capacity with noise covariance  $\mathbf{Q}_{\text{cand}}$  equals the MIMO MAC sum rate capacity,  $\mathbf{Q}_{\text{cand}}$  is a worst case noise for this problem. Now, let us use the method for obtaining the worst case noise introduced in the proof of Theorem 3. Using SDPSOL we get

$$\mathbf{A}_0 = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \quad \text{and} \quad \lambda_0 = 2/3.$$

From (26) we can construct the worst case noise covariance to be

$$\mathbf{Q}_0 = \begin{bmatrix} \mathbf{I} & \frac{\mathbf{H}_1 \mathbf{A}_0 \mathbf{H}_2^\dagger}{\lambda_0} & \frac{\mathbf{H}_1 \mathbf{A}_0 \mathbf{H}_3^\dagger}{\lambda_0} \\ \frac{\mathbf{H}_2 \mathbf{A}_0 \mathbf{H}_1^\dagger}{\lambda_0} & \mathbf{I} & \frac{\mathbf{H}_2 \mathbf{A}_0 \mathbf{H}_3^\dagger}{\lambda_0} \\ \frac{\mathbf{H}_3 \mathbf{A}_0 \mathbf{H}_1^\dagger}{\lambda_0} & \frac{\mathbf{H}_3 \mathbf{A}_0 \mathbf{H}_2^\dagger}{\lambda_0} & \mathbf{I} \end{bmatrix} \quad (40)$$

$$= \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}. \quad (41)$$

We immediately notice that  $\mathbf{Q}_{\text{cand}} = \mathbf{Q}_0$ . Thus, for this case, the singular noise covariance constructed from (26) actually is a worst case noise covariance, even though we are not guaranteed this when the noise is singular. It may, in fact, be that the noise covariance constructed using (26) is *always* a worst case noise covariance (and not just when it is nonsingular), but we have not been able to prove this.

## VII. CONCLUSION

In this paper, we established a duality relationship between two seemingly unrelated regions: the achievable region of the MIMO BC obtained using the dirty paper coding and the capacity region of the MIMO MAC. This duality allows us to easily find the dirty paper achievable region and the dirty paper covariance matrices which achieve the boundary of this region. Though the capacity region of the MIMO BC is unknown due to its nondegraded nature, we were able to show that the dirty paper achievable region achieves the sum rate capacity of the MIMO BC through the use of the MAC-BC duality and the Sato upper bound. These results open up the possibility that the dirty paper region is the actual capacity region of the MIMO BC and also the possibility that other instances of duality exist in Gaussian multiterminal networks.

## APPENDIX

### A. A Covariance Matrix for Flipped Channel

Given a covariance matrix  $\mathbf{\Sigma}$  for some channel  $\mathbf{H}$ , we wish to find a covariance matrix  $\bar{\mathbf{\Sigma}}$  such that  $\text{Tr}(\bar{\mathbf{\Sigma}}) \leq \text{Tr}(\mathbf{\Sigma})$  and

$$\log |\mathbf{I} + \mathbf{H} \mathbf{\Sigma} \mathbf{H}^\dagger| = \log |\mathbf{I} + \mathbf{H}^\dagger \bar{\mathbf{\Sigma}} \mathbf{H}|. \quad (42)$$

If the SVD of  $\mathbf{H}$  is  $\mathbf{H} = \mathbf{F} \mathbf{\Lambda} \mathbf{G}^\dagger$ , where  $\mathbf{\Lambda}$  is square and diagonal, then we propose  $\bar{\mathbf{\Sigma}} = \mathbf{F} \mathbf{G}^\dagger \mathbf{\Sigma} \mathbf{G} \mathbf{F}^\dagger$ . Using the identity

$|\mathbf{I} + \mathbf{A}\mathbf{B}| = |\mathbf{I} + \mathbf{B}\mathbf{A}|$  and the fact that  $\mathbf{F}^\dagger \mathbf{F} = \mathbf{I}$  and  $\mathbf{G}^\dagger \mathbf{G} = \mathbf{I}$ , we can write the capacity of the unflipped channel as

$$\begin{aligned} \log |\mathbf{I} + \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}^\dagger| &= \log |\mathbf{I} + \mathbf{F}\boldsymbol{\Lambda}\mathbf{G}^\dagger\boldsymbol{\Sigma}\mathbf{G}\boldsymbol{\Lambda}\mathbf{F}^\dagger| \\ &= \log |\mathbf{I} + \boldsymbol{\Lambda}\mathbf{G}^\dagger\boldsymbol{\Sigma}\mathbf{G}\boldsymbol{\Lambda}|. \end{aligned} \quad (43)$$

We can similarly write the capacity of the flipped channel with our candidate covariance matrix  $\bar{\boldsymbol{\Sigma}}$  as

$$\begin{aligned} \log |\mathbf{I} + \mathbf{H}^\dagger\bar{\boldsymbol{\Sigma}}\mathbf{H}| &= \log |\mathbf{I} + \mathbf{G}\boldsymbol{\Lambda}\mathbf{F}^\dagger\mathbf{F}\mathbf{G}^\dagger\boldsymbol{\Sigma}\mathbf{G}\mathbf{F}^\dagger\mathbf{F}\boldsymbol{\Lambda}\mathbf{G}^\dagger| \\ &= \log |\mathbf{I} + \mathbf{G}\boldsymbol{\Lambda}\mathbf{G}^\dagger\boldsymbol{\Sigma}\mathbf{G}\boldsymbol{\Lambda}\mathbf{G}^\dagger| \\ &= \log |\mathbf{I} + \boldsymbol{\Lambda}\mathbf{G}^\dagger\boldsymbol{\Sigma}\mathbf{G}\boldsymbol{\Lambda}|. \end{aligned} \quad (44)$$

Therefore, the rate achieved by  $\bar{\boldsymbol{\Sigma}}$  in the flipped channel is the same as the rate achieved by the original covariance matrix  $\boldsymbol{\Sigma}$  in the unflipped channel. It thus only remains to show that  $\text{Tr}(\bar{\boldsymbol{\Sigma}}) \leq \text{Tr}(\boldsymbol{\Sigma})$ . To do so, we make use of the identity  $\text{Tr}(\mathbf{A}\mathbf{B}) = \text{Tr}(\mathbf{B}\mathbf{A})$ . Clearly, we can write

$$\text{Tr}(\bar{\boldsymbol{\Sigma}}) = \text{Tr}(\mathbf{F}\mathbf{G}^\dagger\boldsymbol{\Sigma}\mathbf{G}\mathbf{F}^\dagger) = \text{Tr}(\mathbf{G}^\dagger\boldsymbol{\Sigma}\mathbf{G}) = \text{Tr}(\boldsymbol{\Sigma}\mathbf{G}\mathbf{G}^\dagger). \quad (45)$$

We then use Gram–Schmidt to expand  $\mathbf{G}$  into a full unitary matrix

$$\tilde{\mathbf{G}} = [\mathbf{G}\bar{\mathbf{G}}] \quad (46)$$

such that  $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^\dagger = \tilde{\mathbf{G}}^\dagger\tilde{\mathbf{G}} = \mathbf{I}$ . Using this unitary matrix, we can write

$$\begin{aligned} \text{Tr}(\boldsymbol{\Sigma}) &= \text{Tr}(\boldsymbol{\Sigma}\tilde{\mathbf{G}}\tilde{\mathbf{G}}^\dagger) \\ &= \text{Tr}(\boldsymbol{\Sigma}\mathbf{G}\mathbf{G}^\dagger) + \text{Tr}(\boldsymbol{\Sigma}\bar{\mathbf{G}}\bar{\mathbf{G}}^\dagger) \\ &\geq \text{Tr}(\boldsymbol{\Sigma}\mathbf{G}\mathbf{G}^\dagger). \end{aligned} \quad (47)$$

To get (47), we used the fact that the matrix  $\bar{\mathbf{G}}^\dagger\boldsymbol{\Sigma}\bar{\mathbf{G}}$  is positive semidefinite, which implies  $\text{Tr}(\boldsymbol{\Sigma}\bar{\mathbf{G}}\bar{\mathbf{G}}^\dagger) = \text{Tr}(\bar{\mathbf{G}}^\dagger\boldsymbol{\Sigma}\bar{\mathbf{G}}) \geq 0$ .

### B. Proof of Trace Constraint for Transformations

In this section, we show that the MAC–BC transformations obtained in (8)–(10) satisfy the sum trace requirement. First, we compute

$$\begin{aligned} \text{Tr}(\boldsymbol{\Sigma}_K) &\leq \text{Tr}(\mathbf{A}_K\mathbf{P}_K) \\ &= \text{Tr}(\mathbf{P}_K) + \sum_{i=1}^{K-1} \text{Tr}(\boldsymbol{\Sigma}_i\mathbf{H}_K^\dagger\mathbf{P}_K\mathbf{H}_K). \end{aligned}$$

By adding  $\sum_{i=1}^{K-1} \text{Tr}(\boldsymbol{\Sigma}_i)$  to both sides we get

$$\sum_{i=1}^K \text{Tr}(\boldsymbol{\Sigma}_i) \leq \text{Tr}(\mathbf{P}_K) + \sum_{i=1}^{K-1} \text{Tr}(\boldsymbol{\Sigma}_i(\mathbf{I} + \mathbf{H}_K^\dagger\mathbf{P}_K\mathbf{H}_K)).$$

By the definition of  $\boldsymbol{\Sigma}_j$ , we get

$$\begin{aligned} \text{Tr} \left( \boldsymbol{\Sigma}_j \left( \mathbf{I} + \sum_{i=j+1}^K \mathbf{H}_i^\dagger\mathbf{P}_i\mathbf{H}_i \right) \right) &= \text{Tr}(\boldsymbol{\Sigma}_j\mathbf{B}_j) \\ &\leq \text{Tr}(\mathbf{A}_j\mathbf{P}_j) \\ &= \text{Tr}(\mathbf{P}_j) \\ &\quad + \sum_{i=1}^{j-1} \text{Tr}(\boldsymbol{\Sigma}_i\mathbf{H}_j^\dagger\mathbf{P}_j\mathbf{H}_j). \end{aligned}$$

Using this expression for  $j = K - 1$  we get

$$\begin{aligned} \sum_{i=1}^K \text{Tr}(\boldsymbol{\Sigma}_i) &\leq \sum_{l=K-1}^K \text{Tr}(\mathbf{P}_l) \\ &\quad + \sum_{i=1}^{K-2} \text{Tr} \left( \boldsymbol{\Sigma}_i \left( \mathbf{I} + \sum_{l=K-1}^K \mathbf{H}_l^\dagger\mathbf{P}_l\mathbf{H}_l \right) \right). \end{aligned}$$

By induction, we can further show that

$$\begin{aligned} \sum_{i=1}^K \text{Tr}(\boldsymbol{\Sigma}_i) &\leq \sum_{l=j}^K \text{Tr}(\mathbf{P}_l) \\ &\quad + \sum_{i=1}^{j-1} \text{Tr} \left( \boldsymbol{\Sigma}_i \left( \mathbf{I} + \sum_{l=j}^K \mathbf{H}_l^\dagger\mathbf{P}_l\mathbf{H}_l \right) \right), \end{aligned}$$

for any  $j$ . For  $j = 1$ , we get

$$\sum_{i=1}^K \text{Tr}(\boldsymbol{\Sigma}_i) \leq \sum_{l=1}^K \text{Tr}(\mathbf{P}_l).$$

The same proof method can be used to show that the BC–MAC transformations in (12)–(14) also satisfy the trace constraints.

### C. Finding the Lagrangian Dual Problem

We first find the dual problem of the maximum sumrate of the MIMO MAC<sup>2</sup>

$$\max_{\mathbf{P}_i \in \mathcal{S}} \log \left| \mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger\mathbf{P}_i\mathbf{H}_i \right| \quad (48)$$

over the convex set

$$\mathcal{S} = \left\{ \mathbf{P}_i : \mathbf{P}_i \geq 0 \ \forall i, \sum_{i=1}^K \text{Tr}(\mathbf{P}_i) \leq P \right\}.$$

The problem given by (48) is a convex optimization problem, i.e., it has a concave objective function and a convex constraint set. Hence, a convex Lagrangian dual minimization problem can be obtained that achieves the same optimum value at (48). For this, we rewrite (48) as

$$\begin{aligned} \min_{\mathbf{X}, \mathbf{P}_i} & -\log |\mathbf{X}| \\ \text{such that} & \mathbf{X} = \mathbf{I} + \sum_{i=1}^K \mathbf{H}_i^\dagger\mathbf{P}_i\mathbf{H}_i \end{aligned}$$

$$\sum_{i=1}^K \text{Tr}(\mathbf{P}_i) \leq P, \ \mathbf{P}_i \geq 0.$$

Note that matrix inequalities are associated with dual variables that are matrices, while scalar inequalities are associated with scalar dual variables. The Lagrangian for this problem is

$$\begin{aligned} \mathcal{L}(\mathbf{X}, \mathbf{P}_i, \mathbf{A}, \mathbf{S}_i, \lambda) &= -\log |\mathbf{X}| + \text{Tr} \left[ \mathbf{A}(\mathbf{X} - \mathbf{I} - \sum_{i=1}^K \mathbf{H}_i^\dagger\mathbf{P}_i\mathbf{H}_i) \right] \\ &\quad + \lambda \left( \sum_{i=1}^K \text{Tr}(\mathbf{P}_i) - P \right) + \sum_{i=1}^K \text{Tr}(\mathbf{S}_i\mathbf{P}_i). \end{aligned} \quad (49)$$

<sup>2</sup>This derivation is based on the dual problem found in [4]

The dual function is found by minimizing with respect to the primal variables  $\mathbf{X}, \mathbf{P}_i$

$$g(\mathbf{A}, \mathbf{S}_i, \lambda) = \inf_{\mathbf{X}, \mathbf{P}_i} \mathcal{L}(\mathbf{X}, \mathbf{P}_i, \mathbf{A}, \mathbf{S}_i, \lambda). \quad (50)$$

We obtain the optimality conditions by differentiating the Lagrangian (49) with respect to the primal variables to get

$$\begin{aligned} \lambda \mathbf{I} &= \mathbf{H}_i \mathbf{A} \mathbf{H}_i^\dagger + \mathbf{S}_i, \quad \forall i \\ \mathbf{X}^{-1} &= \mathbf{A}. \end{aligned}$$

For any Lagrangians not satisfying these conditions, we get  $g(\mathbf{A}, \mathbf{S}_i, \lambda) = -\infty$ . For Lagrangians that do satisfy these conditions, we get

$$g(\mathbf{A}, \mathbf{S}_i, \lambda) = \log |\mathbf{A}| - \text{Tr}(\mathbf{A}) - \lambda P + t. \quad (51)$$

The dual problem is then obtained by maximizing the dual function  $g(\mathbf{A}, \mathbf{S}_i, \lambda)$  with respect to the dual variables

$$\begin{aligned} &\max_{\mathbf{A}, \lambda} \log |\mathbf{A}| - \text{Tr}(\mathbf{A}) - \lambda P + t \\ \text{such that } &\mathbf{A} \geq 0, \lambda \geq 0 \\ &\lambda \mathbf{I} \geq \mathbf{H}_i \mathbf{A} \mathbf{H}_i^\dagger, \quad \forall i. \end{aligned} \quad (52)$$

Due to the convexity of the original optimization, the sum rate capacity of the MAC is equal to the solution of the dual problem above (52).

We now use Lagrangian duality to find an alternative form for the Sato upper bound. The Sato upper bound is originally defined in (18) as

$$C_{\text{Sato}}(P, \mathbf{H}) \triangleq \inf_{\mathbf{Z} \in \mathcal{S}} \max_{\Sigma \geq 0, \text{Tr}(\Sigma) \leq P} \log |\mathbf{I} + \mathbf{Z}^{-1} \mathbf{H} \Sigma \mathbf{H}^\dagger|. \quad (53)$$

We will find an alternate form for this upper bound by considering the Lagrangian dual of only the inner maximization (and not of the entire expression). For some fixed  $\mathbf{Z} > 0$ , consider the inner maximization

$$\max_{\Sigma \geq 0, \text{Tr}(\Sigma) \leq P} \log |\mathbf{I} + \mathbf{Z}^{-1} \mathbf{H} \Sigma \mathbf{H}^\dagger|. \quad (54)$$

Note that the maximization in (54) is the capacity of a multiple-antenna system with a channel given by  $\mathbf{Z}^{-1/2} \mathbf{H}$  and additive white Gaussian noise [2]. The capacity of a system with the conjugate transpose channel  $\mathbf{H}^\dagger \mathbf{Z}^{-1/2}$  can be shown to be exactly equal to that in (48) (see [2] for proof). Therefore, (54) is equivalent to

$$\max_{\mathbf{Q} \geq 0, \text{Tr}(\mathbf{Q}) \leq P} \log |\mathbf{I} + \mathbf{H}^\dagger \mathbf{Z}^{-1/2} \mathbf{Q} \mathbf{Z}^{-1/2} \mathbf{H}|. \quad (55)$$

Notice that this maximization is equivalent to the form of the MAC sumrate maximization in (48) with  $K = 1$ ,  $\mathbf{H}_1^\dagger = \mathbf{H}^\dagger \mathbf{Z}^{-1/2}$ , and  $\mathbf{P}_1 = \mathbf{Q}$ . Therefore, the maximization (55) is equivalent to

$$\begin{aligned} &\max_{\mathbf{A}, \lambda} \log |\mathbf{A}| - \text{Tr}(\mathbf{A}) - \lambda P + t \\ \text{such that } &\mathbf{A} \geq 0, \lambda \geq 0 \\ &\lambda \mathbf{I} \geq \mathbf{Z}^{-1/2} \mathbf{H} \mathbf{A} \mathbf{H}^\dagger \mathbf{Z}^{-1/2}. \end{aligned} \quad (56)$$

Clearly, we can multiply the last inequality by the term  $\mathbf{Z}^{-1/2}$  on both sides and negate the objective function to get

$$\begin{aligned} &\min_{\mathbf{A}, \lambda} -\log |\mathbf{A}| + \text{Tr}(\mathbf{A}) + \lambda P - t \\ \text{such that } &\mathbf{A} \geq 0, \lambda \geq 0 \\ &\lambda \mathbf{Z} \geq \mathbf{H} \mathbf{A} \mathbf{H}^\dagger. \end{aligned} \quad (57)$$

Since this is equivalent to the inner maximization of the Sato upper bound for every  $\mathbf{Z} \in \mathcal{S}$ , we can rewrite the Sato upper bound as

$$\begin{aligned} C_{\text{Sato}}(P, \mathbf{H}) &= \inf_{\mathbf{Z} \in \mathcal{S}} \min_{\mathbf{A}, \lambda} -\log |\mathbf{A}| + \text{Tr}(\mathbf{A}) + \lambda P - t \\ &\text{such that} \\ &\lambda \geq 0, \mathbf{A} \geq 0 \\ &\lambda \mathbf{Z} \geq \mathbf{H} \mathbf{A} \mathbf{H}^\dagger. \end{aligned} \quad (58)$$

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