

Optimized Training and Feedback for MIMO Downlink Channels

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Abstract—We consider a MIMO fading broadcast channel where channel state information is acquired at user terminals via downlink training and channel feedback is used to provide transmitter channel state information (CSIT) to the base station. The feedback channel (the corresponding uplink) is modeled as an AWGN channel, orthogonal across users. The total bandwidth consumed is the sum of the bandwidth/resources used for downlink training, channel feedback, and data transmission. Assuming that the channel follows a block fading model and that zero-forcing beamforming is used, we optimize the net achievable rate for unquantized (analog) and quantized (digital) channel feedback. The optimal number of downlink training pilots is seen to be essentially the same for both feedback techniques, but digital feedback is shown to provide a larger net rate than analog feedback.

I. INTRODUCTION

We consider a MIMO Gaussian broadcast channel modeling the downlink of a system where a Base Station (BS) has N_t antennas and serves K single-antenna User Terminals (UTs). A channel use of such system is described by

$$y_k = \mathbf{h}_k^H \mathbf{x} + z_k, \quad k = 1, \dots, K \quad (1)$$

where y_k is the channel output at UT k , $z_k \sim \mathcal{CN}(0, N_0)$ is the corresponding Additive White Gaussian Noise (AWGN), $\mathbf{h}_k \in \mathbb{C}_t^{N_t}$ is the vector of channel coefficients from the BS antenna array to the k -th UT antenna and \mathbf{x} is the vector of channel input symbols transmitted by the BS, subject to the average power constraint $\mathbb{E}[|\mathbf{x}|^2] \leq P$. We use ρ to denote the nominal SNR: $\rho \triangleq \frac{P}{N_0}$. We assume a block fading model, i.e., the channel remains constant over a coherence interval of T channel uses. Albeit suboptimal, we focus on zero-forcing (ZF) beamforming with $K = N_t$ users for its analytical tractability.

In order to perform zero-forcing beamforming (or any other multi-user MIMO strategy), the BS must have an accurate estimate of the channel to each UT. Such information is generally acquired in a two-step process: each UT first estimates its own downlink channel during a common downlink training phase, after which each UT transmits its channel estimate over a feedback channel (on the corresponding uplink) to the BS. The rates achievable with zero-forcing beamforming depend critically on the quality of the CSI available to the BS, but high quality CSI can only be achieved if (a) each UT is able to accurately estimate its own channel (i.e., by using a suitably long downlink training phase), and (b) the process of channel

feedback does not add too much additional distortion to the channel information.

In this work, we attempt to determine the optimum fraction of resources that should be dedicated to training and feedback with the criterion being the *net* spectral efficiency achievable on the downlink. Training consumes downlink bandwidth while channel feedback consumes uplink bandwidth. From a high-level system perspective, increasing either training or feedback effectively takes away from the bandwidth available for actual data transmission. The net spectral efficiency is therefore the transmission rate reflecting the overhead due to the downlink bandwidth consumed for training and the uplink bandwidth used for channel feedback.

We utilize our earlier work in which the downlink spectral efficiency was tightly bounded as a function of the amount of training and feedback [1]. This allows the net spectral efficiency per UT, accounting for training/feedback resources, to be accurately lower bounded as

$$R_k \geq \left(1 - \frac{T_1 + T_{fb}}{T}\right) (R^{ZF} - \overline{\Delta R}) \quad (2)$$

where T_1 and T_{fb} are the number of channel symbols used for training and feedback, respectively, per coherence block of length T .¹ The quantity R^{ZF} denotes the rate achievable with ideal CSI while $\overline{\Delta R}$ is the rate gap due to imperfect CSI. This rate gap depends on the particular feedback strategy used, as specified in [1], but is decreasing in both T_1 and T_{fb} . The objective of this paper is to maximize the net spectral efficiency with respect to T_1 and T_{fb} for the cases of analog and digital feedback, and to understand how the optimal values of T_1 and T_{fb} as well as the optimized net spectral efficiency depend on various system parameters of interest (e.g., blocklength T , signal-to-noise ratio, and N_t).

The present work is an extension of [2], where the same optimization was investigated for the case of analog (unquantized) feedback over a shared MIMO MAC feedback channel. On the other hand, here we consider both analog and digital feedback techniques and focus primarily on an AWGN model for the feedback channel. Interested readers can refer to [2] for a discussion of other prior work on this general topic.

¹Using T_1 and T_{fb} symbols per length T block is fully equivalent to using a fraction $\frac{T_1}{T}$ and $\frac{T_{fb}}{T}$ of the total bandwidth for training and feedback.

II. CHANNEL STATE ESTIMATION AND FEEDBACK

In this section we describe the basic training and channel feedback scheme that allows CSI to be acquired.

1) Common downlink training: T_1 shared pilot symbols (essentially $\frac{T_1}{N_t}$ pilots per BS antenna) are transmitted to allow all UTs to estimate their downlink channel vectors $\{\mathbf{h}_k\}$ based on the observation

$$\mathbf{s}_k = \sqrt{\frac{T_1 P}{N_t}} \mathbf{h}_k + \mathbf{z}_k \quad (3)$$

where $\mathbf{z}_k \sim \mathcal{CN}(0, N_0 \mathbf{I})$. Each UT performs linear MMSE of \mathbf{h}_k from the observation \mathbf{s}_k , which results in a per-coefficient estimation error with variance [1, Equation 7]

$$\frac{1}{1 + \left(\frac{T_1}{N_t}\right) \rho}. \quad (4)$$

2) Channel feedback: Each UT feeds back its channel estimation immediately after the training phase. We focus on the scenario where the feedback channel is modeled as an AWGN channel with the same signal-to-noise ratio ρ , identical to the nominal downlink SNR. Because UT's are assumed to access the feedback channel orthogonally, a total of T_{fb} channel symbols translates into $\frac{T_{\text{fb}}}{N_t}$ feedback channel uses per UT. The different feedback strategies are described in Section III.

From the feedback received from each of the UT', the BS obtains the channel estimate $\hat{\mathbf{h}}_1, \dots, \hat{\mathbf{h}}_{N_t}$. The imperfection in the CSI available to the BS stems from two sources: the channel estimation error during the common training phase, and the distortion incurred during the feedback phase. For analog feedback the distortion is due to additive noise in the feedback channel, while for digital feedback it consists of the quantization error as well as possible errors while transmitting bits over the feedback channel.

If the beamforming vectors $\hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_{N_t}$ are selected by using zero-forcing on the basis of the imperfect channel estimates $\hat{\mathbf{h}}_1, \dots, \hat{\mathbf{h}}_{N_t}$, the following per-UT rate is achievable if equal-power (across UT's) Gaussian inputs are used:

$$\mathbb{E} \left[\log \left(1 + \frac{|\mathbf{h}_k^H \hat{\mathbf{v}}_k|^2 \frac{\rho}{N_t}}{1 + \frac{\rho}{N_t} \sum_{j \neq k} |\mathbf{h}_k^H \hat{\mathbf{v}}_j|^2} \right) \right], \quad (5)$$

assuming each UT is aware of its received SINR.² Imperfect CSI results in non-zero interference coefficients $|\mathbf{h}_k^H \hat{\mathbf{v}}_j|$, which in turn decrease the rate. In [1] it is shown that the rate in (5) is accurately lower-bounded by

$$R^{\text{ZF}} - \overline{\Delta R} \quad (6)$$

where R^{ZF} is the rate achievable with perfect CSI and $\overline{\Delta R}$ denotes the rate gap given by

$$\overline{\Delta R} \triangleq \log \left(1 + \frac{\rho}{N_t} \sum_{j \neq k} \mathbb{E} [|\mathbf{h}_k^H \hat{\mathbf{v}}_j|^2] \right). \quad (7)$$

²Such knowledge can be acquired through an additional dedicated training round as discussed in [1]. This training round does not significantly affect the present work, and thus is ignored for the sake of simplicity.

The rate gap depends on T_1 , T_{fb} and the feedback strategy. Its closed-form expressions are found in [1] for the cases addressed in this paper.

III. OPTIMIZING TRAINING AND FEEDBACK

We now consider the problem of interest, which is the maximization of the *net* spectral efficiency:

$$\max_{T_1, T_{\text{fb}}: T_1 + T_{\text{fb}} \leq T} \left(1 - \frac{T_1 + T_{\text{fb}}}{T} \right) (R^{\text{ZF}} - \overline{\Delta R}(T_1, T_{\text{fb}})). \quad (8)$$

To facilitate solving this optimization, it is useful to write our problem as follows:

$$\max_{T_t \leq T} \max_{T_1 + T_{\text{fb}} = T_t} \left(1 - \frac{T_1 + T_{\text{fb}}}{T} \right) (R^{\text{ZF}} - \overline{\Delta R}(T_1, T_{\text{fb}})). \quad (9)$$

Furthermore, we write the rate gap as follows:

$$\overline{\Delta R}(T_1, T_{\text{fb}}) = \log(1 + g(T_1, T_{\text{fb}})) \quad (10)$$

where the function $g(\cdot)$ depends on the feedback strategy and is defined later. Because the first multiplicative term is constant when $T_1 + T_{\text{fb}} = T_t$, the inner maximization corresponds to minimization of the function $g(\cdot)$ subject to the constraint $T_1 + T_{\text{fb}} \leq T_t$:

$$g(T_t) \triangleq \min_{T_1 + T_{\text{fb}} \leq T_t} g(T_1, T_{\text{fb}}), \quad (11)$$

while the second step is a maximization of the net spectral efficiency over T_t (the total training and feedback symbols):

$$\max_{T_t: T_t \leq T} \left(1 - \frac{T_t}{T} \right) (R^{\text{ZF}} - \log(1 + g(T_t))). \quad (12)$$

In the following this two-step strategy is implemented for analog feedback, TDD systems with channel reciprocity, and digital feedback (with and without feedback channel errors).

A. Analog Feedback

We begin by considering unquantized analog feedback, whereby the complex amplitude of each discrete-time feedback symbol is chosen as the UT's estimate of each complex channel coefficient. Because each UT is allowed $\frac{T_{\text{fb}}}{N_t}$ feedback channel uses, this corresponds to $\frac{T_{\text{fb}}}{N_t^2}$ feedback channel uses per channel coefficient (if $T_{\text{fb}} > N_t^2$, each coefficient is effectively repeated $\frac{T_{\text{fb}}}{N_t^2}$ times on the feedback channel). This results in distortion that is inversely proportional to $\rho \frac{T_{\text{fb}}}{N_t^2}$, and the resulting rate gap is described as [1, Section IV]:

$$g^{\text{analog}}(T_1, T_{\text{fb}}) = \frac{N_t - 1}{T_1} + \frac{N_t(N_t - 1)}{T_{\text{fb}}}. \quad (13)$$

We begin by minimizing $g(\cdot)$ subject to a constraint on $T_1 + T_{\text{fb}}$. For the sake of generality, we rewrite $g(\cdot)$ as:

$$g^{\text{analog}}(T_1, T_{\text{fb}}) = \frac{w_1}{T_1} + \frac{w_{\text{fb}}}{T_{\text{fb}}}. \quad (14)$$

where $w_1 = N_t - 1$ and $w_{\text{fb}} = N_t(N_t - 1)$. Therefore, the minimization to be solved is:

$$\min \quad \frac{w_1}{T_1} + \frac{w_{\text{fb}}}{T_{\text{fb}}} \quad (15)$$

$$\text{subject to} \quad T_1 + T_{\text{fb}} \leq T_t. \quad (16)$$

This is readily seen to be a convex optimization, and can be solved by forming the Lagrangian:

$$\mathcal{L}(T_1, T_{\text{fb}}, \mu) = g(T_1, T_{\text{fb}}) + \frac{1}{\mu^2}(T_1 + T_{\text{fb}})$$

where $\mu > 0$ is the Lagrangian multiplier. The KKT condition yields the following solution

$$T_1^* = \sqrt{w_1}\mu, \quad T_{\text{fb}}^* = \sqrt{w_{\text{fb}}}\mu. \quad (17)$$

In terms of T_t , these can be written as

$$T_1^* = \sqrt{\frac{w_1}{\mathcal{K}}}T_t, \quad T_{\text{fb}}^* = \sqrt{\frac{w_{\text{fb}}}{\mathcal{K}}}T_t \quad (18)$$

where we let $\mathcal{K} = (\sqrt{w_1} + \sqrt{w_{\text{fb}}})^2$, while the objective value is given by

$$g(T_t) = \frac{\mathcal{K}}{T_t} \quad (19)$$

It is clear that T_t is shared between training and feedback proportional to the square root of the weights w_1 and w_{fb} .

Using (19), the overall optimization can now be characterized in terms of a single variable T_t . Namely the second step of the proposed optimization corresponds to maximizing

$$f(T_t) = \left(1 - \frac{T_t}{T}\right) \left[R^{\text{ZF}} - \log\left(1 + \frac{\mathcal{K}}{T_t}\right) \right] \quad (20)$$

Because f is concave in T_t , the optimal T_t^* can be found by numerically solving for $\frac{\partial f}{\partial T_t} = 0$ where

$$\frac{\partial f}{\partial T_t} = \frac{\mathcal{K}\left(1 - \frac{T_t}{T}\right)}{T_t^2\left(1 + \frac{\mathcal{K}}{T_t}\right)} - \frac{1}{T} \left[R^{\text{ZF}} - \log\left(1 + \frac{\mathcal{K}}{T_t}\right) \right]. \quad (21)$$

Although a closed-form solution for T_t^* does not exist, it is possible to compute how this quantity scales with blocklength T . From (21), the optimal T_t^* satisfies the following equality

$$\frac{\mathcal{K}(T - T_t)}{T_t^2\left(1 + \frac{\mathcal{K}}{T_t}\right)} = R^{\text{ZF}} - \log\left(1 + \frac{\mathcal{K}}{T_t}\right) \quad (22)$$

It is easy to see that the derivative in (21) is upperbounded by $\frac{1}{T}\tilde{f}(T_t)$, where

$$\tilde{f}(T_t) = \frac{\mathcal{K}(T - T_t)}{T_t^2} - \left[R^{\text{ZF}} - \frac{\mathcal{K}}{T_t} \right] \quad (23)$$

Since f is concave, it follows that the solution \tilde{T}_t of the equation $\tilde{f}(T_t) = 0$ is an upper bound to the optimal value T_t^* . Solving $f(T_t) = 0$ we find

$$T_t^* \leq \tilde{T}_t = \sqrt{\frac{\mathcal{K}T}{R^{\text{ZF}}}} \quad (24)$$

Furthermore, when the rate gap is small such that $\log\left(1 + \frac{\mathcal{K}}{T_t}\right) \approx \frac{\mathcal{K}}{T_t}$ (which becomes accurate for large T), the upperbound also becomes a very good approximation.

The upperbound (24) yields two interesting behaviors: 1) for a fixed SNR (i.e., constant R^{ZF}) T_t^* increases as $O(\sqrt{T})$ as $T \rightarrow \infty$; 2) for a fixed coherence interval T , T_t^* decreases as $O(1/\sqrt{R^{\text{ZF}}})$ for large SNR, or equivalently, it decreases as

$O(1/\sqrt{\log(\text{SNR})})$ since $R^{\text{ZF}} = \log(\text{SNR}) + O(1)$ for large SNR.

In addition, an upper bound on T_1^* can be reached by combining (24) with (18):

$$\tilde{T}_1 = \sqrt{\frac{w_1}{\mathcal{K}}}\tilde{T}_t = \sqrt{\frac{w_1 T}{R^{\text{ZF}}}} = \sqrt{\frac{(N_t - 1)T}{R^{\text{ZF}}}}. \quad (25)$$

According to this approximation, the optimal downlink training is independent of w_{fb} , and thus of the efficiency of the feedback channel.

Next, we examine the impact of T_t^* on the achievable rate. Using the upperbound (24) into (20), the objective value can be approximated as

$$f(\tilde{T}_t) = \left(1 - \sqrt{\frac{\mathcal{K}}{R^{\text{ZF}}T}}\right) \left[R^{\text{ZF}} - \log\left(1 + \sqrt{\frac{\mathcal{K}R^{\text{ZF}}}{T}}\right) \right]$$

After some manipulation, it can be shown that the resulting effective rate gap with respect to R^{ZF} is given by

$$R^{\text{ZF}} - f(T_t^*) \leq R^{\text{ZF}} - f(\tilde{T}_t) \approx 2\sqrt{\frac{\mathcal{K}R^{\text{ZF}}}{T}} \quad (26)$$

Thus, the gap to a perfect CSI system decreases roughly as $O(1/\sqrt{T})$ as T increases.

B. Time-Division Duplexing

The analysis from the previous subsection can also be used to optimize the amount of uplink training performed in a time-division-duplexed (TDD) system with perfect channel reciprocity (i.e., the downlink and uplink channels are identical).³ Note that no feedback is necessary in such case. In [1, Remark 4.2] the rate gap for a TDD system that uses T_{TDD} uplink training symbols ($\frac{T_{\text{TDD}}}{N_t}$ per MS) is given:

$$\overline{\Delta R} = \log\left(1 + \frac{N_t - 1}{T_{\text{TDD}}}\right). \quad (27)$$

The optimization over T_{TDD} is

$$\max_{T_{\text{TDD}} \leq T} \left(1 - \frac{T_{\text{TDD}}}{T}\right) \left[R^{\text{ZF}} - \log\left(1 + \frac{N_t - 1}{T_{\text{TDD}}}\right) \right], \quad (28)$$

which is clearly equivalent to the optimization in (20) with $T_t = T_{\text{TDD}}$ and $\mathcal{K} = N_t - 1$. As a result, the analysis and approximations from the previous subsection carry over. By adapting (24) we have

$$T_{\text{TDD}}^* \leq \tilde{T}_{\text{TDD}} = \sqrt{\frac{(N_t - 1)T}{R^{\text{ZF}}}}, \quad (29)$$

which is the same as the approximation to T_1^* (the number of downlink training symbols) for analog feedback in (25).

Based upon the expression for the approximate rate gap in (26), by comparing the value of \mathcal{K} for analog feedback and for TDD we see that the rate gap for analog feedback is a factor $1 + \sqrt{N_t}$ larger than for TDD.

For future reference it is also worthwhile to notice that the TDD setting corresponds to the non-TDD setting with perfect

³Note that a similar optimization is considered in [3], although in that work analysis of this optimization is not performed.

feedback (i.e., the BS knows the UT channel estimates, or equivalently $w_{\text{FB}} = 0$ in Section III-A). As a result, the net rate achievable with TDD serves as an upper bound to that achievable with training and channel feedback.

C. Error-Free Digital Feedback

We now analyze digital feedback techniques, whereby each UT quantizes its vector channel estimate to B bits and then maps these bits into $\frac{T_{\text{fb}}}{N_t}$ transmit symbols. For the quantization step we consider a family of random vector quantization (RVQ) schemes. Assuming the feedback bits are received error-free, in [1, Section V] it is shown that the rate gap is

$$\overline{\Delta R} = \log \left(1 + \frac{N_t - 1}{T_1} + \rho 2^{-\frac{B}{N_t - 1}} \right) \quad (30)$$

where the distortion error is expressed in terms of bits. In this section we assume unrealistically that error-free communication is possible over the feedback channel at its underlying capacity of $\log_2 \left(1 + \frac{P}{N_0} \right)$ bits per channel use. Each of the N_t UT's utilize $\frac{T_{\text{fb}}}{N_t}$ channel uses, and therefore $B = \frac{T_{\text{fb}}}{N_t} \log_2 (1 + \rho)$. As a result, we obtain

$$g^{\text{digital}}(T_1, T_{\text{fb}}) = \frac{N_t - 1}{T_1} + \rho (1 + \rho)^{-\frac{T_{\text{fb}}}{N_t(N_t - 1)}}. \quad (31)$$

The first step is the minimization of the above function subject to the constraint $T_1 + T_{\text{fb}} \leq T_t$. Since g^{digital} is convex in T_1, T_{fb} , we form the Lagrangian and readily obtain

$$T_1 = \mu \sqrt{N_t - 1} \quad (32)$$

$$T_{\text{fb}} = N_t(N_t - 1) \frac{2 \ln(\mu) + \ln \left(\frac{\rho \ln(1 + \rho)}{N_t(N_t - 1)} \right)}{\ln(1 + \rho)} \quad (33)$$

Note that the feedback length grows as $O(\ln \mu)$, much slower than the linear increase (in μ) for the common training.

Contrary to the earlier analog feedback case, g^{digital} cannot be expressed as a closed form of T_t but instead must be expressed as a function of μ . However, for the sake of comparison with analog feedback we perform this optimization in terms of T_1 rather than μ . Based upon (32) we can express T_{fb} as a function of T_1 :

$$T_{\text{fb}} = N_t(N_t - 1) \frac{2 \ln(T_1) + \ln \left(\frac{\rho \ln(1 + \rho)}{N_t(N_t - 1)} \right)}{\ln(1 + \rho)}, \quad (34)$$

and thus the net spectral efficiency can be written as:

$$\left(1 - \frac{T_1 + N_t(N_t - 1) \frac{2 \ln(T_1) + \ln \left(\frac{\rho \ln(1 + \rho)}{N_t(N_t - 1)} \right)}{\ln(1 + \rho)}}{T} \right) \times \left[R^{\text{ZF}} - \log \left(1 + \frac{N_t - 1}{T_1} + \frac{N_t(N_t - 1)^2}{(T_1)^2 \ln(1 + \rho)} \right) \right]. \quad (35)$$

Because T_{fb} increases logarithmically with T_1 , its effect on the maximization is rather minimal. As a result, the maximization of T_1 is very similar to the maximization of T_{TDD} in the TDD setting, which is in turn similar to the maximization of T_1 in the presence of analog feedback.

D. Digital Feedback with Errors

Rather than assuming (unrealistically) that the feedback channel operates at channel capacity and error-free, in this section we analyze a system where uncoded QAM is used to transmit each UT's quantized channel vector over the feedback channel. Each UT utilizes $\frac{T_{\text{fb}}}{N_t}$ feedback channel uses. Assuming that quantization bits are arbitrarily mapped to channel symbols, one or more symbol errors (among the $\frac{T_{\text{fb}}}{N_t}$ channel uses) makes the feedback from a particular UT effectively useless and thus leads to a rate effectively of zero. Under this assumption, the achievable net rate is given by:

$$\left(1 - \frac{T_1 + T_{\text{fb}}}{T} \right) (1 - \overline{P}_{e,\text{fb}}) [R_k^{\text{ZF}} - \overline{\Delta R}] \quad (36)$$

where $\overline{\Delta R}$ is defined in (30). Because each UT utilizes $\frac{T_{\text{fb}}}{N_t}$ complex channel symbols, the number of feedback bits per user $B = \frac{T_{\text{fb}}}{N_t} \log_2 M$ where M is the number of constellation points. The per-symbol QAM error probability is given by

$$P_s = 1 - \left(1 - 2 \left(1 - \frac{1}{\sqrt{M}} \right) Q \left(\frac{3(P/N_0)}{M - 1} \right) \right)^2, \quad (37)$$

while the probability of a feedback error, $\overline{P}_{e,\text{fb}}$, is the probability that any of the symbols are received incorrectly:

$$\overline{P}_{e,\text{fb}} = 1 - (1 - P_s)^{\frac{T_{\text{fb}}}{N_t}}. \quad (38)$$

In order to allow for a two-step optimization, we rewrite the objective in (36) as:

$$\left(1 - \frac{T_t}{T} \right) [R_k^{\text{ZF}} - h(T_t)] \quad (39)$$

where the effective rate-loss $h(T_t)$ incorporates the loss due to feedback error is defined as:

$$h(T_t) = \min_{T_1, T_{\text{fb}}: T_1 + T_{\text{fb}} \leq T_t} w(T_1, T_{\text{fb}}) \quad (40)$$

with

$$w(T_1, T_{\text{fb}}) = (1 - \overline{P}_{e,\text{fb}}) \log \left(1 + \frac{N_t - 1}{T_1} + \rho M^{-\frac{T_{\text{fb}}}{N_t(N_t - 1)}} \right) + \overline{P}_{e,\text{fb}} R_k^{\text{ZF}}. \quad (41)$$

If a reasonable constellation size is used, the probability of feedback error is quite small even when the number of feedback bits per users is relatively large (e.g., for $N_t = 4$ at 10 dB with $B = 25$ and 4-QAM, $\overline{P}_{e,\text{fb}} = 0.038$). As a result, the minimization of $w(T_1, T_{\text{fb}})$ is very similar to the minimization of $g^{\text{digital}}(T_1, T_{\text{fb}})$ for error-free feedback in (31), but with a constellation of size M rather than $1 + \rho$. When computed numerically, an optimization over the constellation size is also performed.

IV. NUMERICAL EXAMPLES & DISCUSSION

This section provides some numerical examples to illustrate the analysis of the previous section. The system parameters, unless otherwise specified, are $N_t = 4$ and $\rho = 10$ dB. In all cases the previously stated optimizations have been numerically computed with T_1 and T_{fb} restricted to integers,

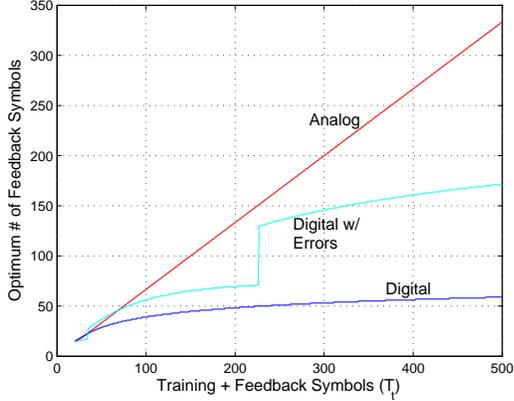


Fig. 1. Optimum number of feedback symbols (T_{fb}) versus total training + feedback (T_t).

subject to the constraints $T_1 \geq N_t$ (to ensure at least one training symbol per channel coefficient) and $T_{fb} \geq N_t^2$ for analog feedback (one feedback symbol per channel coefficient) and $T_{fb} \geq N_t$ for digital feedback (one FB symbol per UT).

In Fig. 1 the optimum number of feedback symbols (T_{fb}) is plotted versus T_t , the total training & feedback budget, for analog feedback, digital feedback, and digital feedback with errors (uncoded QAM). For analog feedback the number of feedback symbols grows linearly with T_t with slope $T_{fb} = \frac{\sqrt{N_t}}{1+\sqrt{N_t}}$, while for digital feedback T_{fb} increases with T_t at a much slower rate (approximately logarithmically). The rate of increase for T_{fb} is particularly slow beyond $T_t = 100$. At this point digital feedback without errors corresponds to 10 symbols per user and thus nearly 35 bits (B). At this point the distortion due to quantization ($2^{-\frac{B}{N_t-1}}$) is less than 10^{-3} and the gains in increasing B beyond this point are very negligible. Even when feedback using uncoded 4-QAM is considered, each user is quantizing to 28 bits at $T_t = 100$. The abrupt shift for digital feedback with errors occurs when the constellation changes from 4-QAM to BPSK: when the number of feedback symbols becomes too large (when T_t is sufficiently large) the probability of feedback error becomes significant and it becomes more efficient to reduce this error probability by reducing the constellation to BPSK while keeping the number of bits (B) nearly the same. This is a consequence of using uncoded transmission on the FB channel.

In Fig. 2 the optimal values of T_1 and T_{fb} are plotted versus blocklength T for analog, digital, and digital w/ errors; T_{TDD} is also plotted for TDD. Most striking is the fact that the optimal values of T_1 and the optimal T_{TDD} are essentially identical for the three feedback techniques as well as for TDD. Furthermore, although not shown here, the optimizing values of T_1 are very well approximated by $\sqrt{\frac{(N_t-1)T}{R_{ZF}}}$ as in (25). On the other hand, the number of feedback symbols depends critically on the feedback method. Because analog feedback is so inefficient, a large number of feedback symbols are used so that the rate gap due to feedback is not too large. On the

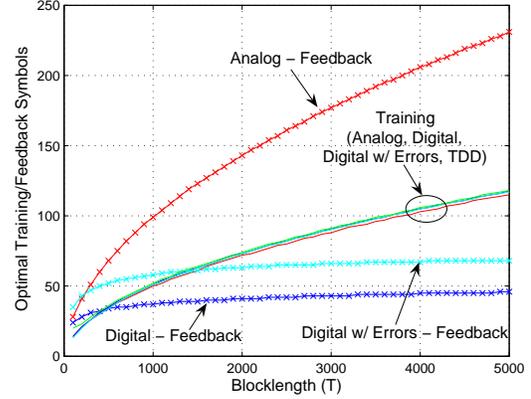


Fig. 2. Optimum number of feedback symbols (T_{fb}) and training symbols (T_1) versus blocklength (T).

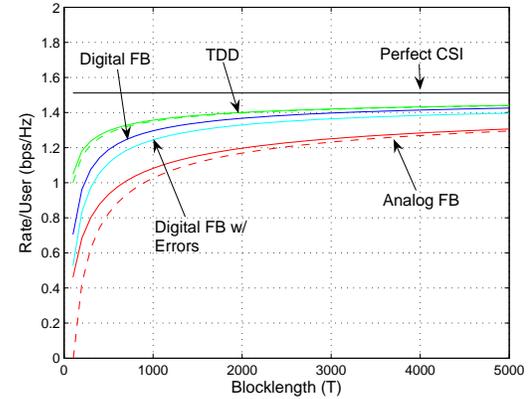


Fig. 3. Achievable sum rate vs. blocklength (T).

other hand, digital feedback is very efficient and a relatively small number of feedback symbols is required.

In Fig. 3 the net achievable rate is plotted versus blocklength T . For analog and TDD the rate approximations based upon (26) are indicated with dotted lines and are seen to become increasingly accurate as T is increased. Analog feedback is outperformed by digital feedback, with or without errors, for all blocklengths. This is because digital feedback offers a significantly smaller distortion as compared to analog whenever T_{fb} is larger than (approximately) N_t^2 (i.e., one symbol per channel coefficient) [1, Section VI], and for reasonable blocklengths it is optimal to use T_{fb} larger than N_t^2 (Fig. 2).

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