On the Required Accuracy of Transmitter Channel State Information in Multiple Antenna Broadcast Channels

Giuseppe Caire
University of Southern California
Los Angeles, CA, USA
Email: caire@usc.edu

Nihar Jindal
University of Minnesota
Minneapolis, MN, USA
Email: nihar@umn.edu

Shlomo Shamai
Technion - IIT
Technion City, Haifa, Israel
Email: sshlomo@ee.technion.ac.il

***Abstract*— A multiple antenna broadcast channel (multiple transmit antennas, one antenna at each receiver) with imperfect channel state information available to the transmitter is considered. If perfect channel state information is available to the transmitter, then a multiplexing gain equal to the minimum of the number of transmit antennas and the number of receivers is achievable. On the other hand, if each receiver has identical fading statistics and the transmitter has no channel information, the maximum achievable multiplexing gain is only one. The focus of this paper is on determination of necessary and sufficient conditions on the rate at which CSIT quality must improve with SNR in order for full multiplexing gain to be achievable. The main result of the paper shows that scaling CSIT quality such that the CSIT error is dominated by the inverse of the SNR is both necessary and sufficient to achieve the full multiplexing gain as well as a bounded rate offset (i.e., the sum rate has no negative sub-logarithmic terms) in the compound channel setting.**

I. Introduction

We consider a multi-input multi-output (MIMO) Gaussian broadcast channel modeling the downlink of a system where the base station (transmitter) has M antennas and M user terminals (receivers) have one antenna each. A channel use of such channel is described by

$$y_k = \mathbf{h}_k^{\mathsf{H}} \mathbf{x} + z_k, \quad k = 1, \ldots, K \tag{1}$$

where y_k is the channel output at receiver k, $z_k \sim \mathcal{CN}(0, N_0)$ is the corresponding AWGN, $\mathbf{h}_k \in \mathbb{C}^M$ is the vector of channel coefficients from the k-th receiver to the transmitter antenna array and $\mathbf{x}$ is the channel input vector. The channel input is subject to the average power constraint $\mathbb{E}[|\mathbf{x}|^2] \leq P$.

If the transmitter has perfect CSIT, dirty paper coding achieves the capacity region of this channel and a multiplexing gain of M is achievable, although each receiver has only one antenna [1–5]. On the other extreme, if each receiver has perfect CSI but the transmitter has no CSIT and each user has identical identical fading statistics (e.g., spatially white Rayleigh fading) then TDMA is optimal and the maximum multiplexing gain is one [1][6]. CSIT is clearly critical for the MIMO broadcast channel. Note that the MIMO broadcast is very different from a point-to-point MIMO channel, in which case the level of CSI available at the transmitter does not affect the multiplexing gain, which is equal to the minimum of the number of transmit and receive antennas, assuming perfect CSI at the receiver and a full rank channel matrix.

Motivated by the extreme cases of perfect and no CSIT, in this paper we consider the following simple question: how accurate must the CSIT be in order for the full multiplexing gain to be achievable? We do not limit ourselves to any particular transmission scheme (e.g., zero-forcing beamforming) but instead ask this from a fundamental information theoretic perspective.

There are two earlier results that more precisely set the context of this work. In [7] it is shown that if the transmitter has imperfect CSIT and the quality of the CSIT is fixed relative to the SNR, then the multiplexing gain achievable by any transmission strategy is upper bounded by $\frac{4}{3}$, which is strictly smaller than the factor of 2 achievable with perfect CSIT. This result implies that that CSIT quality must improve with SNR in order for full multiplexing to be achievable, although the rate of improvement is not specified. On the other hand, in [8][9] it is shown that a naive zero-forcing beamforming strategy performed on the basis of imperfect CSIT can achieve the full multiplexing gain if the error in the CSIT scales as $O(SNR^{-1})$. Therefore, $O(SNR^{-1})$ scaling of CSIT error is a sufficient condition for achieving full multiplexing.

The contribution of this work is an information theoretic analysis of the required scaling of CSIT error in the context of a multiple-antenna *compound* broadcast channel. In the *ergodic* setting, the channels are drawn repeatedly (i.e., per channel use or frame) according to a specified distribution and the long-term average rate (with respect to the channel distribution) is considered; this is the setting of all of the previously cited works. In the compound setting, on the other hand, the channels are randomly drawn once from a specified set but then fixed forever, and the relevant metric is the maximum rate that can be achieved regardless of which particular channel realization was chosen. Thus the ergodic setting considers the average rate achievable over the different channel realizations, while the compound setting is concerned with the worst-case rate over the possible channel realizations.

The multiplexing gain of a compound MIMO broadcast channel in which the possible channel realizations are fixed (i.e., are independent of SNR) is analyzed in [10]. In our setting, the quality of the CSIT is determined by the (angular) *spread* of the possible channel realizations, and thus we consider the case where the potential channel realizations vary with SNR (i.e., where the spread decreases with SNR).

Our first result (Theorem 1) shows that a necessary and sufficient condition for full multiplexing gain is that the ratio of the logarithm of the CSIT error to the logarithm of the SNR be no larger than -1. This condition is slightly weaker than an $O(SNR^{-1})$ condition, and this weakness is a consequence of the coarseness of the multiplexing gain metric. In order to remedy this situation, we further insist on a bounded rate offset (i.e., the achievable sum rate cannot have negative sub-logarithmic terms) in addition to the full multiplexing gain (in the sense of the affine approximation to high-SNR capacity proposed in [11]). We then show (Theorem 2) that CSIT error that scales as $O(SNR^{-1})$ is both necessary and sufficient to achieve the full multiplexing gain and a bounded rate offset, as desired.

II. Main Results

We consider a memoryless compound multiple-antenna broadcast channel in which the transmitter has $M > 1$ antennas and there are M receivers with a single antenna each. For simplicity we state our results for the two transmit antenna ($M = 2$), two user channel. In the compound setting, the channel vector of user 1 has J_1 possible instances $\mathbf{h}_1^1, \ldots, \mathbf{h}_1^{J_1}$, while the channel vector of user 2 has J_2 possible instances $\mathbf{h}_2^1, \ldots, \mathbf{h}_2^{J_2}$. The transmitter has perfect knowledge of the J_i possible channel vectors of each user, but is not aware of the actual realization. On the other hand, each receiver is assumed to know the particular realization. The received signal at user k if the channel takes on realization j is:

$$y_k^j = (\mathbf{h}_k^j)^{\mathsf{H}} \mathbf{x} + z_k^j, \tag{2}$$

where the Gaussian noise z_k^j is independent across users and different channel realizations.

The above setting is strongly motivated by a multiple antenna broadcast channel in which the transmitter receives quantized (digital) channel feedback from each receiver [8][9][12]. In this limited-feedback setting, each receiver learns its own channel vector (presumably through downlink training) and then quantizes this vector according to some quantization codebook and feeds back the index of the quantization. Assuming error-free feedback, the transmitter knows the Voronoi region in which the channel vector lies, but not the actual realization. Although a Voronoi region will typically contain an uncountable number of vectors, it will soon become apparent that it is sufficient to consider the case where the set of possible channel realizations is finite.

Although the capacity region of this generally non-degraded compound broadcast channel is not known, we are able to derive strong results regarding the multiplexing gain of such a system, which is defined as the maximum of

$$\lim_{P \to \infty} \frac{R_1(P) + R_2(P)}{\log_2 P}$$

where $(R_1(P), R_2(P))$ denote achievable rate pairs for power constraint P and the maximum is taken over all achievable rate pairs. In order to derive our results, we make the following assumptions:

- Each of the channel vectors has unit norm, i.e., $||\mathbf{h}_k^j|| = 1$.
- For any i, j , vectors $(\mathbf{h}_1^i, \mathbf{h}_2^j)$ are linearly independent.

Intuitively speaking, accurate CSIT corresponds to a small angular spread between the J_k possible channel realizations. In terms of the quantized channel feedback scenario described earlier, this corresponds to a small Voronoi region and thus to fine/high-rate quantization of the channel vector. Thus, the essence of this paper is determining the rate at which the angular spread of the channel realizations must decrease with SNR in order for full multiplexing to be achievable.

Due to the worst-case nature of the compound channel setting, removing potential channel realizations (i.e., decreasing either J_1 or J_2) cannot decrease capacity, and in fact we can concentrate on the simple case where $J_1 = 1$ (i.e., the channel to receiver one is fixed and known perfectly) and $J_2 = 2$. For notational simplicity, we refer to the two possible realizations of user 2's channel as $\mathbf{h}_{2a}$ and $\mathbf{h}_{2b}$. Although it is possible to consider the general scenario where the vectors $\mathbf{h}_1$, $\mathbf{h}_{2a}$, and $\mathbf{h}_{2b}$ vary with SNR P, the essence of the problem is captured by assuming that vectors $\mathbf{h}_1$ and $\mathbf{h}_{2a}$ are fixed (for all SNR) and only $\mathbf{h}_{2b}$ varies with SNR.

Given these assumptions, we are able to find a necessary and sufficient condition for achievability of the full multiplexing gain.

Theorem 1: The following is a necessary and sufficient condition for achieving the full multiplexing gain of two in the two-user, two transmit antenna compound broadcast channel under the assumptions $||\mathbf{h}_1|| = ||\mathbf{h}_{2a}|| = ||\mathbf{h}_{2b}|| = 1$ with channel vectors $\mathbf{h}_1$ and $\mathbf{h}_{2a}$ fixed for all SNR's:

$$\lim_{P \to \infty} \frac{\log\left(1 - |\mathbf{h}_{2a}^{\mathsf{H}} \mathbf{h}_{2b}|^2\right)}{\log P} \leq -1. \tag{3}$$

Proof: (Sufficiency) We first prove sufficiency by showing that full multiplexing can be achieved with simple zero-forcing beamforming. The input is chosen as $\mathbf{x} = \mathbf{v}_1 u_1 + \mathbf{v}_2 u_2$ where u_1, u_2 are i.i.d. zero-mean complex Gaussian's, each with variance $\frac{P}{2}$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit norm vectors chosen orthogonal to $\mathbf{h}_{2a}$ and $\mathbf{h}_1$, respectively. The rates $R_1 = I(U_1; Y_1)$ and $R_2 = \min(I(U_2; Y_{2a}), I(U_2; Y_{2b}))$ are achievable, where the mutual information expressions are given by:

$$\begin{aligned}
I(U_1;Y_1) &= \log_2\left(1+\frac{P}{2}|\mathbf{h}_1^{\mathsf{H}}\mathbf{v}_1|^2\right)\\
I(U_2;Y_{2a}) &= \log_2\left(1+\frac{P}{2}|\mathbf{h}_{2a}^{\mathsf{H}}\mathbf{v}_2|^2\right)\\
I(U_2;Y_{2b}) &= \log_2\left(1+\frac{\frac{P}{2}|\mathbf{h}_{2b}^{\mathsf{H}}\mathbf{v}_2|^2}{N_0+\frac{P}{2}|\mathbf{h}_{2b}^{\mathsf{H}}\mathbf{v}_1|^2}\right)\\
&= \log_2\left(1+\frac{\frac{P}{2}|\mathbf{h}_{2b}^{\mathsf{H}}\mathbf{v}_2|^2}{N_0+\frac{P}{2}\left(1-|\mathbf{h}_{2b}^{\mathsf{H}}\mathbf{h}_{2a}|^2\right)}\right)
\end{aligned}$$

Because $|\mathbf{h}_1^{\mathsf{H}}\mathbf{v}_1|^2$ and $|\mathbf{h}_{2a}^{\mathsf{H}}\mathbf{v}_2|^2$ are positive constants, the quantities $I(U_1;Y_1)$ and $I(U_2;Y_{2a})$ each have a multiplexing gain of one. The term $I(U_2;Y_{2b})$ can be lower bounded as:

$$\begin{aligned}
I(U_2;Y_{2b}) \;\geq\; & \log_2\left(1+P|\mathbf{h}_{2b}^{\mathsf{H}}\mathbf{v}_2|^2\right)\\
&-\log_2\left(N_0+\frac{P}{2}\left(1-|\mathbf{h}_{2b}^{\mathsf{H}}\mathbf{h}_{2a}|^2\right)\right).
\end{aligned}$$

The condition in (3) implies $|\mathbf{h}_{2b}^{\mathsf{H}}\mathbf{v}_2|^2$ converges to the constant $|\mathbf{h}_{2a}^{\mathsf{H}}\mathbf{v}_2|^2$ and thus the first term in the above upper bound has a multiplexing gain of one. Furthermore, it is straightforward to confirm that the multiplexing gain of the second term is zero if (3) is satisfied. ■

Proof: (Necessity) In order to prove the necessity of the condition in (3), we upper bound the capacity region by giving outputs (Y_{2a},Y_{2b}) to receiver 1 which creates a degraded channel. We then utilize the upper bound on the degraded compound broadcast channel given in [13], which states:

$$R_2 \leq \min\{I(U;Y_{2a}),I(U;Y_{2b})\} \tag{4}$$

$$R_1+R_2 \leq I(\mathbf{X};Y_1,Y_{2a},Y_{2b}|U) + \min\{I(U;Y_{2a}),I(U;Y_{2b})\} \tag{5}$$

for some marginal-preserving joint distribution with the structure $U\to\mathbf{X}\to(Y_1,Y_{2a},Y_{2b})$.

We now show that (3) is a necessary condition for the degraded channel. Because the broadcast channel capacity region depends only on the marginal probabilities, we can assume arbitrary correlation between the additive noise at users $1,2a$, and $2b$; for our upper bound, we find it best to assume these noises are independent.

Applying the chain rule on the first term in (5) gives

$$I(\mathbf{X};Y_1,Y_{2a},Y_{2b}|U) = I(\mathbf{X};Y_{2a},Y_{2b}|U) + I(\mathbf{X};Y_1|U,Y_{2a},Y_{2b}). \tag{6}$$

Using Markovity, the first term in this expansion is upper bounded as:

$$\begin{aligned}
I(\mathbf{X};Y_{2a},Y_{2b}|U) &= I(\mathbf{X};Y_{2a}|U)+I(\mathbf{X};Y_{2b}|U,Y_{2a})\\
&\leq I(\mathbf{X};Y_{2a}|U)+I(\mathbf{X};Y_{2b}|U).
\end{aligned} \tag{7}$$

The second term in (6) can be upper bounded as:

$$\begin{aligned}
I(\mathbf{X};Y_1|U,Y_{2a},Y_{2b}) &= h(Y_1|U,Y_{2a},Y_{2b})-h(Y_1|\mathbf{X})\\
&\leq h(Y_1|Y_{2a},Y_{2b})-h(Y_1|\mathbf{X})\\
&= I(\mathbf{X};Y_1|Y_{2a},Y_{2b}),
\end{aligned} \tag{8}$$

where the first and third lines follow from Markovity (i.e, $h(Y_1|\mathbf{X},U,Y_{2a},Y_{2b}) = h(Y_1|\mathbf{X}) = h(Y_1|\mathbf{X},Y_{2a},Y_{2b}) = h(Z_1)$) and the second line because conditioning reduces entropy. Furthermore, we trivially have:

$$\min\{I(U;Y_{2a}),I(U;Y_{2b})\} \leq \frac{1}{2}\left(I(U;Y_{2a})+I(U;Y_{2b})\right). \tag{9}$$

Using (6), (7), (8), and (9) in the R_1+R_2 upper bound in (5) therefore gives the following sum rate upper bound:

$$\begin{aligned}
R_1+R_2 \leq\; & I(\mathbf{X};Y_{2a}|U)+I(\mathbf{X};Y_{2b}|U)+\\
& I(\mathbf{X};Y_1|Y_{2a},Y_{2b})+\frac{1}{2}\left(I(U;Y_{2a})+I(U;Y_{2b})\right).
\end{aligned} \tag{10}$$

In order to achieve the full multiplexing gain, R_2 must achieve a multiplexing gain of one, which by (4) implies $I(U;Y_{2a})$ and $I(U;Y_{2b})$ each have multiplexing gain one. This, however, implies that the first two terms in (10) have zero multiplexing gain. To see this note

$$\begin{aligned}
I(\mathbf{X};Y_{2a}|U)+I(U;Y_{2a}) &= I(\mathbf{X},U;Y_{2a}) = I(\mathbf{X};Y_{2a})\\
I(\mathbf{X};Y_{2b}|U)+I(U;Y_{2b}) &= I(\mathbf{X},U;Y_{2b}) = I(\mathbf{X};Y_{2b}).
\end{aligned}$$

Since Y_{2a} and Y_{2b} are single antenna outputs, the quantities $I(\mathbf{X};Y_{2a})$ and $I(\mathbf{X};Y_{2b})$ each have at most a multiplexing gain of 1. Thus, if $I(U;Y_{2a})$ and $I(U;Y_{2b})$ each have multiplexing gain 1 then the multiplexing gains of $I(\mathbf{X};Y_{2a}|U)$ and $I(\mathbf{X};Y_{2b}|U)$ are each upper bounded by 0.

As a result, the right hand side of (10) can have multiplexing gain of two only if the term $I(\mathbf{X};Y_1|Y_{2a},Y_{2b})$ has a multiplexing gain of at least one. In other words, the following is necessary condition for full multiplexing:

$$\lim_{P\to\infty}\frac{I(\mathbf{X};Y_1|Y_{2a},Y_{2b})}{\log P} \geq 1. \tag{11}$$

We upper bound $I(\mathbf{X};Y_1|Y_{2a},Y_{2b})$ as follows:

$$\begin{aligned}
I(\mathbf{X};Y_1|Y_{2a},Y_{2b}) &= h(Y_1|Y_{2a},Y_{2b})-h(Y_1|\mathbf{X})\\
&= h(Y_1-f(Y_{2a},Y_{2b})|Y_{2a},Y_{2b})-\log 2\pi e\\
&\leq h(Y_1-f(Y_{2a},Y_{2b}))-\log 2\pi e\\
&\leq \log\mathbb{E}[|Y_1-f(Y_{2a},Y_{2b})|^2]
\end{aligned} \tag{12}$$

where $f(\cdot,\cdot)$ is any mapping $\mathbb{C}^2\to\mathbb{C}$, that may of course depend on $\mathbf{h}_1,\mathbf{h}_{2a},\mathbf{h}_{2b}$.

In order to obtain the tightest bound, we can choose $f(\cdot,\cdot)$ to be the MMSE estimator (or linear MMSE estimator) of Y_1 from (Y_{2a},Y_{2b}). However, since we are interested in the high SNR regime, it is sufficient to let $f(\cdot,\cdot)$ be the least-squares approximation of Y_1, given by

$$f(Y_{2a},Y_{2b}) = \mathbf{h}_1^{\mathsf{H}}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}})^{-1}\mathbf{H}_2\begin{bmatrix} Y_{2a}\\ Y_{2b}\end{bmatrix}$$

where we define the matrix $\mathbf{H}_2 = [\mathbf{h}_{2a},\mathbf{h}_{2b}] \in \mathbb{C}^{2\times 2}$. It follows that

$$Y_1 - f(Y_{2a},Y_{2b}) = -\mathbf{h}_1^{\mathsf{H}}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}})^{-1}\mathbf{H}_2\begin{bmatrix} Z_{2a}\\ Z_{2b}\end{bmatrix}$$

and, eventually,

$$\mathbb{E}[|Y_1-f(Y_{2a},Y_{2b})|^2] = \mathbf{h}_1^{\mathsf{H}}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}})^{-1}\mathbf{h}_1 \tag{13}$$

From the definition of eigenvalues in terms of Rayleigh quotients and using the fact that $||\mathbf{h}_1|| = 1$ we have

$$\mathbf{h}_1^{\mathsf{H}}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}})^{-1}\mathbf{h}_1 \leq \lambda_{\max}((\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}})^{-1}) = \frac{1}{\lambda_{\min}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}})}. \tag{14}$$

The matrix $\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}}$ has the same eigenvalues as $\mathbf{H}_2^{\mathsf{H}}\mathbf{H}_2$, which is given by:

$$\mathbf{H}_2^{\mathsf{H}}\mathbf{H}_2 = \begin{bmatrix} 1 & \rho \\ \rho^H & 1 \end{bmatrix} \tag{15}$$

where $\rho = \mathbf{h}_{2a}^{\mathsf{H}}\mathbf{h}_{2b}$. This matrix has eigenvalues $1 + |\rho|$ and $1 - |\rho|$. Because our sufficient condition is specified in terms of $|\rho|^2$, it is convenient to write this expression in terms of $|\rho|^2$ rather than $|\rho|$. A Taylor expansion of the minimum eigenvalue in terms of $|\rho|^2$ about the point $|\rho|^2 = 1$ gives $\lambda_{\min}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}}) = \frac{1}{2}\left(1 - |\rho|^2\right) + O(|\rho|^4)$. Therefore, (11) translates to the following necessary condition:

$$\lim_{P\to\infty} \frac{\log\left(\frac{1}{1-|\rho|^2}\right)}{\log P} \geq 1, \tag{16}$$

which implies (3). ■

If the quantity $\left(1 - |\mathbf{h}_{2a}^{\mathsf{H}}\mathbf{h}_{2b}|^2\right)$, which can be thought of as the error in the CSIT, scales as $O\left(P^{-1}\right)$ then the condition of Theorem 1 clearly is satisfied. However, the condition of Theorem 1 is in fact looser than $O\left(P^{-1}\right)$ because scaling $\left(1 - |\mathbf{h}_{2a}^{\mathsf{H}}\mathbf{h}_{2b}|^2\right)$ as $\frac{\log P}{P}$ also satisfies (3). As the following simple example shows, this looseness is in fact non-trivial. Consider the following simple set of channels:

$$\mathbf{h}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{h}_{2a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{h}_{2b} = \begin{bmatrix} \epsilon \\ \sqrt{1-\epsilon^2} \end{bmatrix}.$$

The condition in (3) translates to:

$$\lim_{P\to\infty} \frac{\log \epsilon^2}{\log P} \leq -1. \tag{17}$$

The sum rate achievable (using the zero-forcing technique used to show the sufficiency of the condition in Theorem 1) for two valid scalings, $\epsilon^2 = \frac{1}{P}$ and $\epsilon^2 = \frac{\log_2 P}{P}$, (along with the sum rate achieved for $\epsilon^2 = 0$, is shown in Figure 1). Both ϵ^2 scalings achieve a multiplexing gain of two, but the difference between the reference perfect CSIT curve ($\epsilon^2 = 0$) and the rate achieved with $\epsilon^2 = \frac{\log_2 P}{P}$ increases double-logarithmically with SNR and thus is unbounded; on the other hand, the difference between the rate achieved with $\epsilon^2 = 0$ and $\epsilon^2 = \frac{1}{P}$ is bounded for all SNR's. Furthermore, from the figure we see that the double-logarithmic rate loss is non-trivial even at moderate SNR values.

This behavior is due to the coarseness of the multiplexing gain metric, which is only a zero-th order approximation of capacity at high SNR and is unable to capture the effect of sub-polynomial (but non-trivial) terms. In order to remedy this situation, it is necessary to consider the *affine* approximation to high-SNR capacity proposed in [11]:

$$C(P) = \mathcal{S}_\infty \left(\log_2 P - \mathcal{L}_\infty\right) + o(1), \tag{18}$$

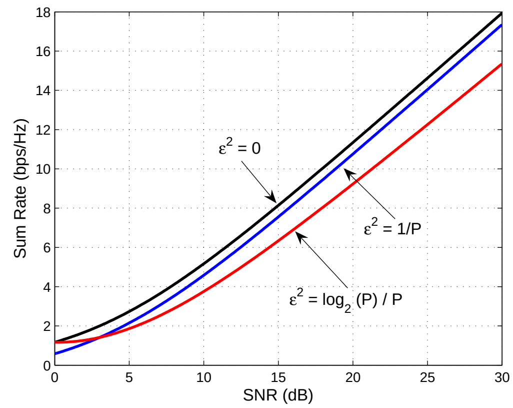

Fig. 1. Achievable Sum Rate for Different CSIT Scaling

where constants $\mathcal{S}_\infty$ and $\mathcal{L}_\infty$ are the multiplexing gain and rate offset, respectively. This affine approximation has been very useful in differentiating between systems with the same multiplexing gain but different rate offsets, e.g., CDMA systems with different receivers [11], point-to-point MIMO systems with spatially correlated fading [14], and MIMO broadcast employing sub-optimal linear-precoding techniques on the basis of perfect CSIT [15].

If we require both full multiplexing gain ($\mathcal{S}_\infty = 2$) and a bounded rate offset (i.e. finite $\mathcal{L}_\infty$), then we can modify the proof of Theorem 1 to show that the condition $O\left(P^{-1}\right)$ is both necessary and sufficient.

Theorem 2: The sum rate capacity of the two-user, two transmit antenna compound broadcast channel (assuming $||\mathbf{h}_1|| = ||\mathbf{h}_{2a}|| = ||\mathbf{h}_{2b}|| = 1$ with channel vectors $\mathbf{h}_1$ and $\mathbf{h}_{2a}$ fixed for all SNR's) admits an affine expansion in the form of (18) with $\mathcal{S}_\infty = 2$) and finite $\mathcal{L}_\infty$ if and only if $\left(1 - |\mathbf{h}_{2a}^{\mathsf{H}}\mathbf{h}_{2b}|^2\right) = O\left(P^{-1}\right)$.

Proof: Sufficiency follows trivially using the same approach as Theorem 1. To prove necessity, we begin at the sum rate capacity bound in (10):

$$\begin{aligned} R_1 + R_2 \quad \leq \quad & I(\mathbf{X}; Y_{2a}|U) + I(\mathbf{X}; Y_{2b}|U) + \\ & I(\mathbf{X}; Y_1|Y_{2a}, Y_{2b}) + \frac{1}{2}\left(I(U; Y_{2a}) + I(U; Y_{2b})\right). \end{aligned} \tag{19}$$

If we assume the sum rate capacity achieves the full multiplexing gain and a bounded rate offset, $R_1(P)$ and $R_2(P)$ must each have a multiplexing gain of one and a bounded rate offset. As a result, $I(U; Y_{2a})$ and $I(U; Y_{2b})$ must each have multiplexing gain one and a bounded rate offset. Because the quantities $I(\mathbf{X}; Y_{2a})$ and $I(\mathbf{X}; Y_{2b})$ have a maximum multiplexing gain of one but cannot have *positive* sub-logarithmic terms (for any choice of input $\mathbf{X}$), the terms $I(\mathbf{X}; Y_{2a}|U)$ and $I(\mathbf{X}; Y_{2b}|U)$ must be $O(1)$. From the bound in (10) this implies that $I(\mathbf{X}; Y_1|Y_{2a}, Y_{2b})$ has a multiplexing gain of one, exactly as in the previous proof, and that the following quantity

$$\log P - \log I(\mathbf{X}; Y_1|Y_{2a}, Y_{2b}) \tag{20}$$

does not go to positive infinity. Since $I(\mathbf{X}; Y_1|Y_{2a}, Y_{2b}) \leq \log \mathbb{E}[|Y_1 - f(Y_{2a}, Y_{2b})|^2] \leq \log \frac{1}{\lambda_{\min}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}})}$, this implies that

$$\log P - \log\left(\frac{1}{\lambda_{\min}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}})}\right) = \log\left(P\lambda_{\min}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}})\right) \quad (21)$$

also does not go to infinity. Due to the monotonicity of the $\log(\cdot)$ function and because $\lambda_{\min}(\mathbf{H}_2\mathbf{H}_2^{\mathsf{H}}) = 1 - |\rho|^2 + O(|\rho|^4)$, this further implies that $P(1 - |\rho|^2)$ does not go to infinity, i.e., $(1 - |\rho|^2) = O\left(P^{-1}\right)$. ■

Theorems 1 and 2 easily extend to the M-transmit antenna, M-receiver broadcast channel. Full multiplexing is achieved if and only if each user achieves a multiplexing gain of one, and thus each pair of users must achieve full multiplexing. Thus, the proofs of Theorems 1 and 2 are applied to a M-antenna, 2 user channel with $J_1 = 1$ and $J_2 = M$ and the same dependence on the behavior of the minimum eigenvalue of matrix $\mathbf{H}_2$ can be shown.

The main result can also be nicely interpreted in terms of limited channel feedback systems. As mentioned earlier, in such systems each receiver quantizes the direction of its channel vector to B bits according to some vector quantization codebook and feeds these bits back to the transmitter. In an ideal (but not necessarily achievable) B-bit vector quantization codebook, each of the 2^B Voronoi regions would be a spherical cap of area 2^{-B}. Some simple geometry confirms that the $\sin^2$ of the angle between the center of such a cap and a vector on its boundary (note that $\sin^2(\mathbf{w}, \mathbf{v}) = 1 - |\mathbf{w}^{\mathsf{H}}\mathbf{v}|^2$ for unit norm vectors $\mathbf{w}, \mathbf{v}$) is equal to $2^{-\frac{B}{M-1}}$. Furthermore, it is straightforward to use the results of [16] to show existence of actual vector quantization codebooks such that the $\sin^2$ of the center and boundary of the largest Voronoi region is within a constant factor of $2^{-\frac{B}{M-1}}$. In order to ensure that the quantity $2^{-\frac{B}{M-1}}$ decreases at least as fast as the P^{-1} condition of Theorem 2, it is necessary for B to be proportional to $(M-1)\log_2 P$, which agrees with the sufficiency results in [8].

III. Conclusion

In this work we have derived necessary and sufficient conditions on the rate at which CSIT quality must increase with SNR for achievability of the full multiplexing gain and a bounded rate offset for the multiple antenna broadcast channel in the compound setting. This result indicates the fundamental necessity of scaling CSIT quality with SNR in multi-user MIMO downlink systems and exactly matches with prior work showing that full multiplexing is achievable using simple zero-forcing beamforming strategies if CSIT is appropriately scaled.

Although the compound setting, which essentially considers the worst-case rate achievable over the set of all possible channel realizations, appears to capture the essence of the problem at hand, it remains to rigorously show that the same necessary condition also applies to the less stringent ergodic setting.

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