Symmetric Capacity of MIMO Downlink Channels

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***Abstract*— This paper studies the symmetric capacity of the MIMO downlink channel, which is defined to be the maximum rate that can be allocated to every receiver in the system. The symmetric capacity represents absolute fairness and is an important metric for slowly fading channels in which users have symmetric rate demands. An efficient and provably convergent algorithm for computing the symmetric capacity is proposed, and it is shown that a simple modification of the algorithm can be used to compute the minimum power required to meet given downlink rate demands. In addition, the difference between the symmetric and sum capacity, termed the fairness penalty, is studied. Exact analytical results for the fairness penalty at high SNR are provided for the 2 user downlink channel, and numerical results are given for channels with more users.**

I. Introduction

The multiple antenna downlink (or broadcast) channel has been the subject of much research recently, primarily because impressive multiple antenna capacity benefits can be realized without requiring large numbers of mobile antennas (c.f. [1]). Most research has concentrated on the sum capacity, or maximum total throughput, of this channel. While this is an extremely useful metric, the resulting rate allocation to receivers can be very non-uniform: users with strong channels are typically allocated more rate than users with weaker channels, and some users may not be allocated any rate at all. This may be undesirable in certain systems, particularly those with unequal receiver channel qualities, and a more attractive option is to allocate rates more uniformly. Thus, the *symmetric capacity*, defined to be the maximum rate that can be allocated to every receiver [2], is an important capacity metric.

We focus on quasi-static channels, where the channel is fixed over the time period of interest (i.e., over the period of the delay constraints). In this scenario, the instantaneous rates achievable during a particular channel realization are of importance because no scheduling over different channel realizations can be performed. The symmetric capacity represents the fairest rate allocation in such a scenario. Note that the majority of work on scheduling and fairness for downlink channels (e.g., research on the proportionally-fair algorithm) concentrates on long-term average rates. We alternatively focus on instantaneous rates, which are meaningful when mobility is limited and delay constraints are very stringent.

In this paper we develop an algorithm that computes the symmetric capacity of a multiple-input multiple-output (MIMO) downlink channel. While efficient algorithms for computing the sum capacity as well as the boundary of the capacity region exist [3][4], these cannot be used to directly compute the symmetric capacity, which is the point on the boundary of the capacity region that intersects with the 45 degree line through the origin (i.e., $R_1 = \cdots = R_K$). We characterize the symmetric capacity as a convex program and then utilize the ellipsoid method to find the symmetric capacity.

A simple modification of the algorithm can be used to compute the intersection of the capacity region boundary and any arbitrary ray from the origin, e.g., find where the line $R_2 = 2R_1$ intersects the capacity boundary. This is of interest if differentiated service is to be provided to different sets of users: for example, a set of premium users may be guaranteed double the rate of non-premium users. This modified algorithm can also be used to compute the minimum power required to achieved a desired rate vector. This is clearly of use to operators who need to determine the power required to meet specified rate demands. Though this paper focuses on the MIMO downlink channel, the proposed algorithms can be applied to essentially any convex capacity region, e.g., the MIMO uplink channel as well as fading and/or wideband uplink or downlink channels.

We recently found a similar work in [5] for a wireline channel environment, referred to as *balanced capacity* which corresponds to our differentiated service capacity. Their work considers single antenna frequency selective broadcast and multiple access channels, and the capacity is found by iteratively solving Karush-Kuhn-Tucker (KKT) conditions. However, this algorithm does not easily extend to the multiple antenna channel considered here.

In addition, we study the difference between the sum capacity and the symmetric capacity, which represents the penalty for requiring absolutely fair rate allocation, and is thus termed the *fairness penalty*. The fairness penalty is exactly quantified in the limit of high signal-to-noise ratio (SNR) for a 2 user channel, and numerical results are provided for systems with more users. We find that this fairness penalty is quite small, which indicates that requiring fair rate allocation induces very little reduction in total system throughput. Note that a similar conclusion has been drawn for downlink channels with an asymptotically large number of users and a fixed number of antennas [6].

II. System Model & Background

We consider a K user Gaussian MIMO downlink channel in which the transmitter has M antennas and each receiver has N antennas. The received signal vector $\mathbf{y}_k$ for user k is given by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{n}_k \qquad k = 1, \cdots, K, \tag{1}$$

where $\mathbf{H}_k (\in \mathbb{C}^{N\times M})$ is the channel gain matrix for user k, $\mathbf{x}$ is the transmit signal vector having a power constraint $\mathrm{tr}(E[\mathbf{x}\mathbf{x}^H]) \leq P$, and $\mathbf{n}_k$ $(k = 1, \cdots, K)$ is complex Gaussian noise with unit variance per vector component (i.e., $E[\mathbf{n}_k^H \mathbf{n}_k] = \mathbf{I}$). We assume that the transmitter has perfect knowledge of all channel matrices and each receiver has perfect knowledge of its own channel matrix. Note that we only consider static, or fixed, channels.

It is now well known that dirty paper coding achieves the capacity region of the MIMO downlink channel [7], which we denote as $\mathcal{C}(\mathbf{H}, P)$. We include the variable $\mathbf{H}$, which refers to the aggregate channel from the transmitter to all mobiles, to emphasize the fact that $\mathcal{C}(\mathbf{H}, P)$ is the capacity for a specific channel realization. While the rate equations describing the capacity region are somewhat intractable, the dual MIMO uplink channel, which has the same capacity region as the MIMO downlink channel [8], allows for a simpler characterization of the capacity region. This dual characterization allows the boundary of the capacity region $\mathcal{C}(\mathbf{H}, P)$ to be found by solving the following convex optimization problem:

$$f(\boldsymbol{\mu}) \triangleq \max \boldsymbol{\mu} \cdot \mathbf{R} \;\text{ subject to }\; \mathbf{R} \in \mathcal{C}(\mathbf{H}, P), \tag{2}$$

where $\boldsymbol{\mu} = [\mu_1 \; \mu_2 \; \cdots \; \mu_K]^T$ is the rate reward vector which will be referred to as the *weight* vector and $\mathbf{R} = [R_1 \; R_2 \; \cdots \; R_K]^T$ is the rate vector. The optimization finds the point on the boundary of the capacity region where the tangent to the capacity region is defined by the weight vector. A steepest-descent algorithm is proposed in [4] to solve this maximization for any weight vector. By solving (2) for all possible weight vectors satisfying $\sum_{i=1}^{K} \mu_i = 1$, the entire boundary of the capacity region can be traced out.

The *symmetric capacity* is defined as the maximum of the minimum of all user rates:

$$C^{\mathrm{sym}}(\mathbf{H}, P) \triangleq \max_{\mathbf{R}\in\mathcal{C}(\mathbf{H},\mathbf{P})} \min(R_1, \ldots, R_K). \tag{3}$$

The structure of $\mathcal{C}(\mathbf{H}, P)$ clearly implies that the rate vector achieving the symmetric capacity is the point where $\mathcal{C}(\mathbf{H}, P)$ intersects the line defined by $R_1 = R_2 = \cdots = R_K$.

Though the capacity region $\mathcal{C}(\mathbf{H}, P)$ can be found using (2), this does not directly give the symmetric capacity because the weight vector $\boldsymbol{\mu}$ corresponding to the rate vector achieving the symmetric capacity (i.e., the slope of the boundary of the capacity region at the symmetric capacity point) depends on the channel realization. The sum capacity corresponds to setting all weights to be equal (i.e., $\mu_k = 1/K$ $(k = 1, 2, \cdots, K)$), but does not necessarily give symmetric rates. Fig. 1 illustrates the characterization of the capacity region boundary in terms of the weights for a two-user case. By varying μ_1 and μ_2, different points on the boundary at the region can be found. Note that the weights specify the slope at the tangent to the boundary.

Notations: Boldface letters denote matrix-vector quantities and $(\cdot)^T$ and $(\cdot)^H$ represent transpose and Hermitian transpose, respectively. For element-wise operations, the ith element of a vector $\mathbf{v}$ is defined as $[\mathbf{v}]_i$ or v_i and the sub-vector from i to j $(j \geq i)$ elements of a vector $\mathbf{v}$ as $[\mathbf{v}]_i^j$.

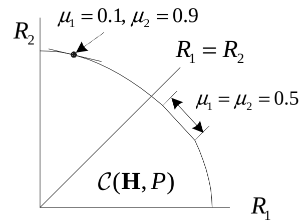

Fig. 1. Characterization of the boundary of the capacity region in terms of the weight vector

III. Capacity Algorithms

In this section, algorithms computing the symmetric capacity and differentiated capacity as well as a method to find the minimum required power to achieve a given rate vector are described.

A. Symmetric Capacity

It is theoretically feasible to directly solve for the symmetric capacity using the dual multiple access channel (MAC) characterization of $\mathcal{C}(\mathbf{H}, P)$, but this would result in a convex optimization problem with more than 2^K constraints, which is impractical for even moderate values of K. As an alternative, we develop an iterative algorithm that finds the slope of the boundary of the capacity region at the symmetric capacity rate vector, and by doing so finds the symmetric capacity. We first show that the symmetric capacity can be characterized as a simple convex program:

Theorem 1: The symmetric capacity is equal to the minimum weighted sum of rates, where the minimum is taken over all possible weight vectors summing to one:

$$\begin{aligned} C^{\mathrm{sym}}(\mathbf{H}, P) &= \min_{\boldsymbol{\mu}:\boldsymbol{\mu}\geq 0,\; \sum_{i=1}^{K}\mu_i = 1} f(\boldsymbol{\mu}) &(4)\\ &= \min_{\mu_1,\ldots,\mu_{K-1}} h(\mu_1, \ldots, \mu_{K-1}), &(5)\end{aligned}$$

where $f(\boldsymbol{\mu})$ is defined in (2) and $h(\mu_1, \ldots, \mu_{K-1})$ is defined as:

$$h(\mu_1, \ldots, \mu_{K-1}) \triangleq f\left(\mu_1, \ldots, \mu_{K-1}, 1 - \sum_{i=1}^{K-1} \mu_i\right).$$

Furthermore, both $f(\boldsymbol{\mu})$ and $h(\mu_1, \ldots, \mu_{K-1})$ are convex functions and thus can be efficiently minimized.

Proof: See [9]. ■

This theorem simplifies the problem of finding the symmetric capacity to a $K-1$ dimensional unconstrained convex program, for which efficient techniques exist. Note that the function $f(\boldsymbol{\mu})$, and thus $h(\cdot)$, can be computed using the algorithm in [4]. Since $f(\boldsymbol{\mu})$ and $h(\cdot)$ are defined as maximums, it is not clear if they are differentiable. However, a subgradient to $h(\cdot)$ can be found: if $\tilde{\mathbf{R}} \in \mathcal{C}(\mathbf{H}, P)$ achieves $h(\tilde{\mu}_1, \ldots, \tilde{\mu}_{K-1})$, then it is straightforward to show that $\mathbf{s}$ with $s_i = \tilde{\mathbf{R}}_i - \tilde{R}_K$ for $i = 1, \ldots, K-1$ is a subgradient.

In order to solve the convex minimization in (5) we utilize the ellipsoid algorithm, which can be used if subgradients can be computed and which provably converges to the optimum. The ellipsoid algorithm is essentially a generalization of the bisection method to multi-dimensional space (see [10] for more details) and is directly applied to the minimization of $h(\mu_1, \ldots, \mu_{K-1})$ in (5), as detailed in Algorithm 1. Note that variables $\mathbf{s}$ and $\mathbf{x}$ are $(K-1)$-dimensional vectors, $\mathbf{E}$ is a $(K-1) \times (K-1)$ matrix, and $\boldsymbol{\mu}$ and $\mathbf{R}$ are K-dimensional vectors.

The algorithm is initiated by designating an ellipse that covers all feasible optimum points ($\mu_i \geq 0$ and $\sum_{i=1}^{K-1} \mu_i \leq 1$), i.e. $\mathbf{E} = (1-1/K)\mathbf{I}_{K-1}$ and $x_i = \mu_i = 1/K$ for $i = 1, \ldots, K-1$ (and $\mu_K = 1/K$). In each step, the rate vector maximizing $\boldsymbol{\mu} \cdot \mathbf{R}$ is found. The weights $\mu_1, \ldots, \mu_K$ are then adjusted to attempt to equalize all user rates: users with large rates have their weights decreases, while users with small rates have their weights increased. The algorithm is terminated either when all rates $R_1,\ R_2,\ \cdots,$ and R_K are sufficiently identical, or when the length of the major axis (i.e. the largest eigenvalue of $\mathbf{E}$) of the ellipsoid is sufficiently small (step 2). If $K = 2$, the ellipsoid method reduces to standard one-dimensional bisection (on μ_1). The MATLAB code implementing Algorithm 1 is available at http://www.ece.umn.edu/users/nihar/symmetric_cap_code.html.

Algorithm 1 Symmetric capacity with ellipsoid algorithm

1. $\mathbf{R} = \arg\max \boldsymbol{\mu} \cdot \mathbf{R}$ subject to $\mathbf{R} \in \mathcal{C}(\mathbf{H}, P)$ (see [4])
2. if $(\max(\text{eig}(\mathbf{E})) < tol)$ or $(|R_k - R_K| < tol, \forall k)$
   break
3. Compute subgradient

$$\mathbf{s} = [\mathbf{R}]_1^{K-1} - R_K \quad (6)$$

$$\tilde{\mathbf{s}} = \frac{\mathbf{s}}{\sqrt{\mathbf{s}^T \mathbf{E} \mathbf{s}}} \quad (7)$$

4. Update ellipse

$$\mathbf{x}^+ = \mathbf{x} - \frac{1}{K}\mathbf{E}\tilde{s} \quad (8)$$

$$\mathbf{E}^+ = \frac{(K-1)^2}{(K-1)^2 - 1}\left(\mathbf{E} - \frac{2}{K}\mathbf{E}\tilde{\mathbf{s}}\tilde{\mathbf{s}}^T\mathbf{E}\right) \quad (9)$$

$$\{\boldsymbol{\mu}^+\}_1^{K-1} = \mathbf{x}^+, \quad \mu_K^+ = 1 - \sum_{i=1}^{K-1} \mu_i^+ \quad (10)$$

A standard proof for the convergence of the ellipsoid algorithm is given in [10], and it is easily shown that $h(\cdot)$ satisfies the required Lipschitz condition. However, the rate vectors returned by the ellipsoid algorithm can behave unusually due to the fact that the boundary of the capacity region has many flat (i.e., linear) sections, which correspond to weights of some subset of users being identical.

When there are two receivers the capacity region boundary is flat at the sum rate plane ($\mu_1 = \mu_2$), as illustrated in Fig. 2. When the symmetric rate vector lies outside of the sum rate plane, as in Fig. 2(a), the algorithm quickly converges to

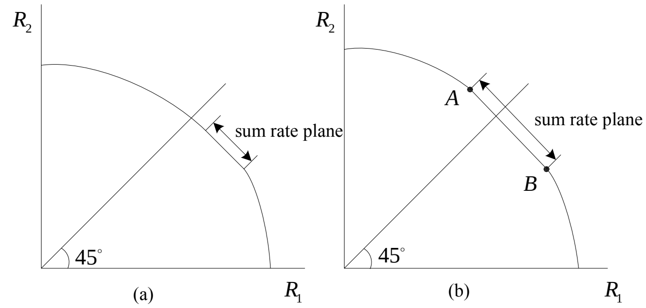

Fig. 2. Capacity region boundary examples for two users

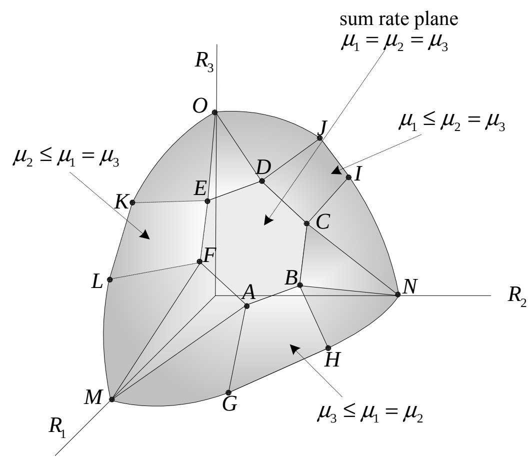

Fig. 3. Capacity region example for a three user case

the weight vector characterizing the symmetric rate vector. In Fig. 2(b), convergence is slightly complicated because the symmetric rate vector lies along the sum rate plane. The ellipsoid method will quickly determine that the optimal weight vector is in the vicinity of $\mu_1 = \mu_2 = 0.5$, but will eventually oscillate between points close to A and B. This is because if μ_1 is slightly smaller than μ_2, the solution to $\max \boldsymbol{\mu} \cdot \mathbf{R}$ is a vector very close to point A, while the solution to the maximization for μ_2 slightly larger than μ_1 is a vector very close to B. When there are more than 2 users, the capacity region boundary has a large number of flat sections, as illustrated in Fig. 3 for a 3 user channel. Thus, oscillations of this sort are even more likely to occur, but note that algorithm still converges to the symmetric capacity because the feasible ellipse continues to decrease in each step.

B. Differentiated Service Capacity

As an alternative to providing equal rate service to all users, it may also be of interest to provide differentiated service to users, in which ratios of the rates allocated to each user are fixed. In a two user channel, as shown in Fig. 4, it may be of interest to compute the largest rate vector such that $R_1 = 2R_2$ (or vice versa). For arbitrary K, finding the differentiated rate vector corresponding to $R_1 = \frac{R_2}{\alpha_2} = \frac{R_3}{\alpha_3} = \cdots = \frac{R_K}{\alpha_K}$ is equivalent to finding the point where the line defined by $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K)$ with $\alpha_1 = 1$ intersects the boundary of the

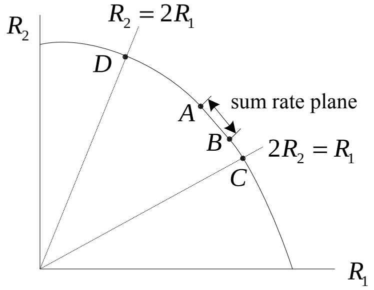

Fig. 4. Differentiated service for 2 user channel

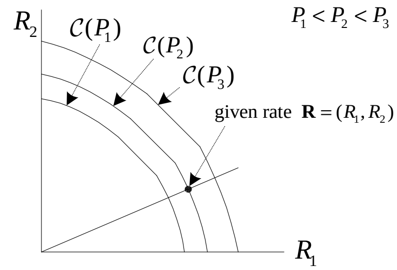

Fig. 5. Boundaries of the capacity region with different powers

capacity region. We denote this as $C^{\text{diff}}(\boldsymbol{\alpha}, \mathbf{H}, P)$, and the following is easily shown using the proof of Theorem 1:

$$C^{\text{diff}}(\boldsymbol{\alpha}, \mathbf{H}, P) = \min_{\boldsymbol{\mu}: \boldsymbol{\mu} \geq 0, \ \sum_{i=1}^{K} \mu_i = 1} \ \max_{\mathbf{R} \in \mathcal{C}(\mathbf{H}, P)} \sum_{i=1}^{K} \frac{\mu_i}{\alpha_i} R_i. \quad (11)$$

As a result, the ellipsoid algorithm can also be used to find $C^{\text{diff}}(\boldsymbol{\alpha}, \mathbf{H}, P)$, with the only changes being that $\sum_{i=1}^{K} \frac{\mu_i}{\alpha_i} R_i$ should be maximized in step 1, the condition in step 2 should check $|\frac{R_k}{\alpha_k} - \frac{R_K}{\alpha_K}|$, and the subgradient in step 3 is given by $\mathbf{s} = \frac{[\mathbf{R}]_1^{K-1}}{[\boldsymbol{\alpha}]_1^{K-1}} - \frac{R_K}{\alpha_K}$.

C. Minimum Power to Achieve a Rate Vector

Another important problem is determining the minimum required transmit power to achieve a given rate vector $\mathbf{R}$. Previous research has provided sub-optimal solutions to this problem [11][12], while we provide the optimal solution. The key observation is that the differentiated capacity algorithm can be used to determine the intersection of the capacity region and an arbitrary line through the origin.

The algorithm can be simply described by considering a two user example, as shown in Fig. 5. Here the minimum required power is P_2, because the $\mathbf{R}$ lies on the boundary of $\mathcal{C}(P_2)$. In addition, the line through the origin and $\mathbf{R}$ is also drawn. The minimum power can be found by exploiting the fact that the intersection of $\mathcal{C}(\mathbf{H}, P)$ and this line can be determined using the differentiated capacity algorithm (with $\alpha_i = \frac{R_i}{R_1}$) described in Section III-B. The differentiated capacity algorithm is run for some initial guess of power such as P_1. The returned rate vector (i.e. the intersection of the line and $\mathcal{C}(P_1)$) is smaller than the desired rate vector, and thus the power must be increased. This process can be continued using the one-dimensional bisection method, until the minimum power is reached. The algorithm is explicitly described in Algorithm 2. Note that initial values for P_{high} and P_{low} can easily be determined [9].

Algorithm 2 Minimum power to achieve a desired rate vector

1. if $P_{\text{high}} - P_{\text{low}} < tol$
   break
2. $P = (P_{\text{high}} + P_{\text{low}})/2$
3. Compute differentiated service capacity: $C^{\text{diff}}(\boldsymbol{\alpha}, \mathbf{H}, P)$
4: If $C^{\text{diff}}(\boldsymbol{\alpha}, \mathbf{H}, P) > R_1$, then $P_{\text{high}} = P$, else $P_{\text{low}} = P$.

IV. Symmetric Capacity vs. Sum Capacity

While symmetric capacity results in absolute fairness amongst all users, it can result in sub-optimal total system throughput. To be more specific, if the symmetric capacity vector does not lie on the sum rate plane, there is a strict reduction in system throughput. This throughput loss is referred to as the *fairness penalty*, i.e., the penalty paid for requiring absolute rate fairness. This penalty will clearly depend on the number of users and antennas, the system SNR, and the specific channel realization.

First notice that the symmetric capacity vs. SNR curve has the same slope, i.e., multiplexing gain, as the sum capacity vs. SNR curve. This is intuitively clear because all spatial degrees of freedom are used when achieving symmetric capacity. However, there can be an absolute rate difference, or equivalently a rightward shift of the capacity curve, between sum capacity and symmetric capacity which is not captured by the multiplexing gain alone. It is often useful to study the limiting behavior of this rate difference at asymptotically high SNR, using the high SNR approximation developed in [13].

We provide exact analytical results for the high SNR fairness penalty for 2 user broadcast channels with a single receive antenna ($N = 1$), and also discuss numerical results for channels with more than two users. In this scenario, the channel matrix can be reduced to vector quantities $\mathbf{h}_k (\in \mathbb{C}^{1 \times M})$. We define the high SNR fairness penalty $\Delta(\mathbf{H})$ as:

$$\Delta(\mathbf{H}) \triangleq \lim_{P \to \infty} \left[C^{\text{sum}}(\mathbf{H}, P) - K \cdot C^{\text{sym}}(\mathbf{H}, P) \right], \quad (12)$$

where $C^{\text{sum}}(\mathbf{H}, P)$ is sum of the rate vector $\mathbf{R}$ in $C(\mathbf{H}, P)$ that maximizes the sum of rates. Although $\Delta(\mathbf{H})$ is defined at asymptotically high SNR, it is generally quite accurate for even moderate SNR values.

Theorem 2: For 2 user downlink channels with $N = 1$ and $M \geq 2$, the fairness penalty is given by:

$$\Delta(\mathbf{H}) = \begin{cases} 0, & \sin^2\theta \leq \frac{\|\mathbf{h}_1\|^2}{\|\mathbf{h}_2\|^2} \leq \frac{1}{\sin^2\theta}, \\ 2\log\left(\frac{\|\mathbf{h}_i\|^2 + \|\mathbf{g}_j\|^2}{2\|\mathbf{h}_i\|\|\mathbf{g}_j\|}\right), & \text{otherwise}, \end{cases} \quad (13)$$

where θ is the angle between channel vectors $\mathbf{h}_1$ and $\mathbf{h}_2$, $\|\mathbf{g}_j\| = \|\mathbf{h}_j\| \, |\sin\theta|$, $i = \arg\min_{k \in \{1,2\}} \{\|\mathbf{h}_k\|\}$, and $j = \arg\max_{k \in \{1,2\}} \{\|\mathbf{h}_k\|\}$.

Proof: See [9]. ■

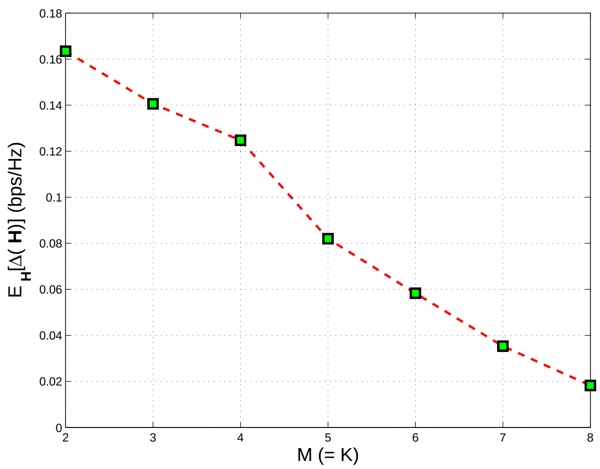

Fig. 6. Expected fairness penalty with respect to the number of transmit antennas M (= number of users K).

For channels that satisfy the condition:

$$\sin^2\theta \leq \frac{\|\mathbf{h}_1\|^2}{\|\mathbf{h}_2\|^2} \leq \frac{1}{\sin^2\theta}, \quad (14)$$

the symmetric capacity is exactly equal to the sum capacity at sufficiently high SNR. Using the notation of Theorem 2, this condition is equivalent to $||\mathbf{g}_j|| \leq ||\mathbf{h}_i||$, where $\mathbf{g}_j$ is the projection of $\mathbf{h}_j$, the larger channel vector, onto the null space of $\mathbf{h}_i$, the smaller vector. This condition is satisfied if the channel norms are sufficiently close, or if the angle between the channel vectors is sufficiently small. The first of these reasons is intuitively clear, since one would expect there to be a penalty associated with allocating equal rates if channel norms are very disparate; the second reason is less intuitive and has to do with the dependence of the shape of the capacity region on the angle θ. When the condition in (14) is not satisfied, the fairness penalty $\Delta(\mathbf{H})$ can be shown to be a monotonically increasing function of $\|\mathbf{g}_j\|$. Thus, the penalty is an increasing function of the disparity between the channel norms and of the angle between the channel vectors.

No similar expression is known for channels with $K > 2$, primarily because it is very difficult to analytically characterize the symmetry capacity at high SNR in this scenario. However, numerical results indicate that the gap is also small in this scenario. In Fig. 6 the expectation (over iid Rayleigh fading channels for each mobile) of the fairness penalty is plotted as a function of the number of transmit antennas M for a downlink channel with $K = M$ and $N = 1$. It is clear from the figure that the fairness penalty is very small and is also decreasing to zero as M (or K) is increased. Notice that we adopt the scenario that both M and K are simultaneously increasing. We have found out that the decrease of the penalty is due to the increase of M.

While Fig. 6 indicates that the fairness penalty is very small for homogeneous users, i.e., users with the same average SNR, the heterogeneous scenario is perhaps of more practical interest. When average SNR's are asymmetric, the fairness penalty is expected to be larger, but preliminary numerical results indicate that this penalty term is again surprisingly small. For example, in a 4 user channel with $M = 4, N = 1$ and average SNR's of 30 dB, 25 dB, 20 dB, and 15 dB, the penalty is calculated to be 2.3 bps/Hz, which translates into a power penalty of less than 2 dB. Our ongoing research efforts are focused on deriving analytical expressions for the fairness penalty for downlink channels with many users and antennas, in both the homogeneous and heteregenous scenarios.

V. Conclusion

We proposed provably convergent algorithms for computing the symmetric capacity of the MIMO downlink channel as well as for computing the minimum required power to achieve a desired rate vector. In fast fading scenarios where users have very loose delay constraints (e.g., file transfer), rate fairness is evaluated in terms of long-term *average* rates. When delay constraints are more stringent and fading occurs on a slow time scale, traffic demands must be met on a very fast time scale, and thus fairness must be evaluated in terms of short-term, or instantaneous, rates. In this setting, the symmetric capacity is a crucial metric for evaluating system performance. In addition, results comparing symmetric and sum capacity show that total system throughput is only slightly decreased if absolute rate fairness is required.

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