Maximizing the Worst-User's Capacity for a Multi-User OFDM Uplink Channel

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Abstract

This project considers the challenges in maximizing the worst-user's rate for a multiple access Gaussian vector channel under power constraints. The multiple access Gaussian vector channel is derived from a multi-user OFDM uplink channel, which has no interference between subchannels. The optimal rate is determined at the maximal equal rate among the users. For a simple treatment, a multi-user scalar channel is considered initially and then a single-user vector channel is considered to show the optimal allocation strategy. For the multi-user vector channel, the solution has yet to be found in this project but one conjectured algorithm is shown.

1 Introduction

Though the concept of multicarrier modulation with non-overlapping subchannels was introduced in the 1950s, it was not aggressively pursued because of its inherent compexity [1]. Later on Weinstein [2] suggested using a DFT to modulate signals with orthogonally overlapping subcarriers. With the advances in VLSI and the development of FFT algorithms, OFDM came into practice in the 1980s. Various standards for wired and wireless systems cam to adopt OFDM in the 1990s, especially when high-rate data transmission was required. Since 2000, it is being applied to cellular environments for the next generation mobile communications.

Basically, OFDM converts a high-rate data stream into multiple low-rate streams, sending them simultaneously over orthogonal subcarriers. Since the required pulse duration for each low-rate stream is relatively long, it is inherently more robust against inter symbol interference (ISI). In addition, introducing a guard interval, which typically adds a cyclic prefix, allows for the complete removal of ISI [1][3]. Another prominent feature of OFDM is the use of FFT techniques to decompose signals into orthogonal subcarriers, allowing for low complexity [2].

There are several ways to accommodate multiusers beyond point-to-point communication. One way, which is a combination of OFDM and CDMA, known as Multi-Carrier(MC)-CDMA, first spreads the signal with a spreading code and then sends it through a number of subcarriers. Another technique involves the splitting of the subcarriers among users, which is equivalent to FDMA, is known as OFDMA. In addition, OFDMA can also incorporate TDMA for users to access the system in a certain timeslot. This technique is called multiuser OFDM [1].

Communication with multiple users through an OFDM channel can be configured in various ways since the channel has many controllable resources. The controllable resources typically include the position of the subcarriers, the number of subcarriers for each user, and the amount of power for each subcarrier. These resources should be allocated in a way to maximize the system performance. Typical measures of performance are aggregate capacity, worst-user's capacity and total transmitted power.

With certain assumptions concerning the channel state, many allocation strategies have been proposed. For the downlink case, maximizing a given rate vector was solved in [4] along with the proof of the duality with the minimizing power to support. This is called the sum-rate capacity. But maximizing the sum-rate capacity has a drawback, when there is very good channel user or very bad channel user. Only the former is allowed to transmit, and there is no chance to transmit for the latter. To resolve this issue, [5] dealt with the proportional fairness among the users. In [6] maximizing the worst-user's rate was addressed but with considerations only for the downlink. It should be noted that every individual user has a common power constraint in downlink while each user has an independent power constraint in uplink. As for the uplink, the structure of the capacity region was characterized as a polymatriod [7] and maximization of the sum-rate capacity was considered in [8][9], with [8], also showing an iterative algorithm to achieve the optimal point.

In this project, I will consider the problem of maximizing the worst-user' rate for an uplink OFDM channel. I will assume that the channel gains are known in advance and that each user has individual power constraint. The channel gains can be estimated at the transmitting end for a time division duplex (TDD) system, but the system model is a frequency division duplex (FDD) because all of the channels are used for the uplink communication. So, for the transmitter to have the channel gain information, it should be feedbacked in practice. There are two specific questions to answer the problem. The first one pertains to the optimal rate for maximizing the worst-user's rate, with the other one being the nature of the allocation strategy.

This project report is organized as follows. Section 2 will describe an uplink OFDM channel, derive a model, and formulate the problem providing assumptions. Section 3 will solve the problem for a simple case (scalar channel). Section 4 will extend for the case of a vector channel and provide an algorithm. Section 5 will conclude with some remarks.

2 Problem Description

A block diagram of multi-user OFDM for an uplink transmission is depicted in Figure 1. Assuming the channel state information (CSI) is known, each user's data is allocated to a certain subcarrier position with a certain amount of power. The calculations for subcarrier and bit allocation are performed at a base station with the results sent back to the corresponding users. According to the number of bits in a subcarrier, adaptive modulation is performed. This is then followed with typical OFDM processing: inverse fast Fourier transformation (IFFT), conversion of the parallel data stream into serial ones and insertion of a guard (typically done with cyclic prefix).

There are many negative characteristics inherent to OFDM caused by the overlapping subcarriers and the use of IFFT/FFT. Since the subcarrier channels are overlapped, it is vulnerable to the interchannel interference when there is slight synchronization error. In addition, the ratio of the instantaneous peak value of the signal at the output of IFFT to its time-averaged value is large

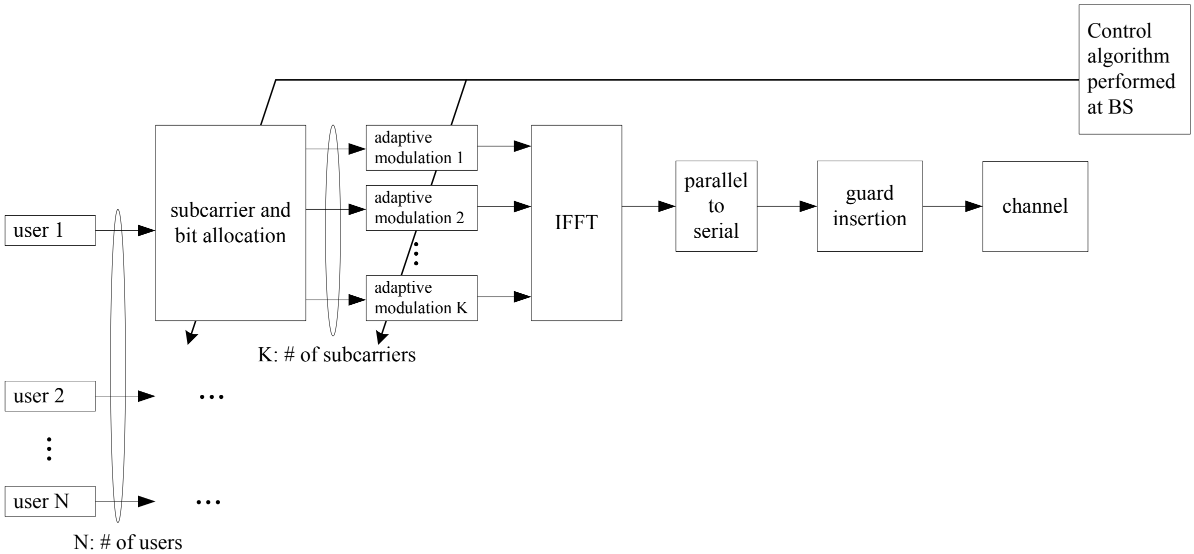

Figure 1: Block diagram of an uplink multi-user OFDM TX system

enough to cause non-linear amplification by the power amp [1]. In this project, however, I will assume an *ideal situation*; i.e., the channel is maintaining orthogonality and there are no nonlinear effects. Thusly, we can model the scheme as a parallel channel assuming the channel has additive Gaussian noise.

Figure 2 shows a model of N-user K-dimension vector Gaussian channel. In this model, X_n $(n = 1, \cdots, N)$ are the channel inputs, $Y_k (k = 1, \cdots, K)$ are the channel outputs and $Z_k (k = 1, \cdots, K)$ are additive Gaussian noises. The Z_k in the model can be interpreted as its variance divided by the fading gain of each channel, assuming fading gain is constant. And $h_{n,k}[\cdot]$ denotes the channel gain for the kth channel of the nth user.

In a matrix vector form, the model can be expressed as

$$\mathbf{Y} = \sum_{n=1}^{N} \mathbf{H}_n \mathbf{X}_n + \mathbf{Z} \tag{1}$$

where $\mathbf{Y}$, $\mathbf{X}_n$, and $\mathbf{Z}$ are $K \times 1$ vectors and $\mathbf{H}_n$ is a diagonal matrix. $\mathbf{H}_n$ is assumed to be known and $\mathbf{Z}$ has a Gaussian distribution with $\mathcal{N}(\mathbf{0}, \mathbf{I})$.

In this project, I will deal with the optimization problem associated with maximizing the worst-user's transmission rate. The *worst-user* is the person who has the lowest maximum transmission rate due to small channel gains and a small power constraint. Note that each user has its own power constraint. Two questions arise: (1) What is the optimal rate for each user? (2) How should we allocate power to achieve this optimal rate in a point of information theory? I will approach this problem from the viewpoint of channel capacity. That is to say that input distributions can be continuous instead of finite, and so no practical digital modulation schemes will be considered. All variables are real-valued, not complex, for a simple deployment. But extension to complex-valued variables is not difficult. Typically, information is packed in a complex-value and

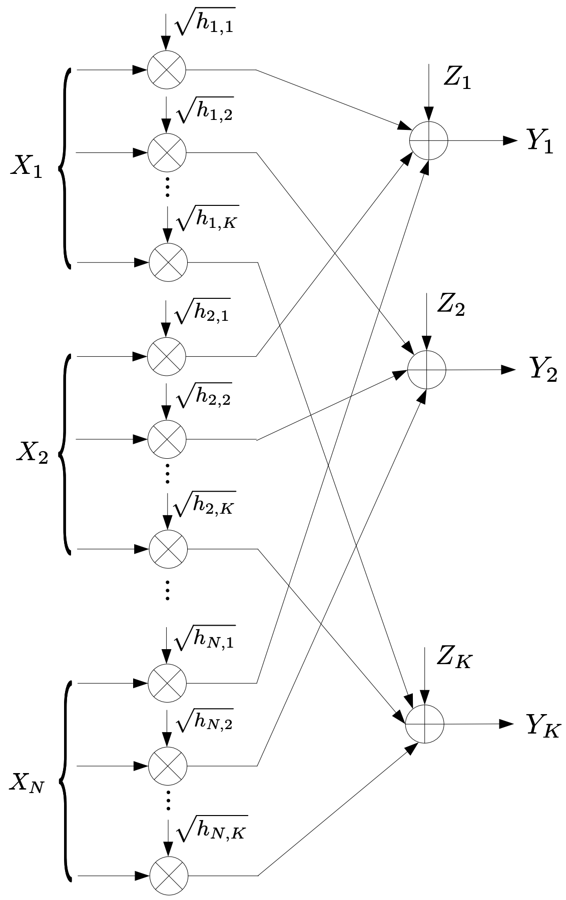

Figure 2: Multi-user vector channel model

transmitted through a complex-valued channel since there are always two orthogonal waveforms at one frequency.

3 Scalar Channel

For simplicity, let us examine the problem for a multi-user scalar channel first. Unlike a vector channel, no allocation strategy is possible. So in this case, I will examine which point in the capacity region is the solution of the problem. Two-user multiple access channel, for example, is given by

$$Y = \sqrt{h_1}X_1 + \sqrt{h_2}X_2 + Z \tag{2}$$

where X_1 and X_2 are two transmitters' random variables, Y is a receiver random variable, h_1 and h_2 are channel gains and Z is an additive Gaussian noise. Without loss of generality (WLOG), Z is assumed to be $\mathcal{N}(0,1)$. Note also that both transmitters have power constraint: $E[|X_1|^2] \leq P_1$ and $E[|X_2|^2] \leq P_2$. Then the capacity region, which is the set of all achievable rate pairs (R_1, R_2), is given as in [10] by the closure of the convex hull of all satisfying

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log_2(1+h_1P_1) \\ R_2 &\leq \frac{1}{2}\log_2(1+h_2P_2) \\ R_1+R_2 &\leq \frac{1}{2}\log_2(1+h_1P_1+h_2P_2) \end{aligned} \tag{3}$$

Now consider the problem of maximizing the worst user's capacity:

$$\begin{aligned} \text{maximize} \quad & \min\{R_1, R_2\} \\ \text{subject to} \quad & R_1 \leq \frac{1}{2}\log_2\Big(1+h_1P_1\Big) \\ & R_2 \leq \frac{1}{2}\log_2\Big(1+h_2P_2\Big) \\ & R_1+R_2 \leq \frac{1}{2}\log_2\Big(1+h_1P_1+h_2P_2\Big) \end{aligned} \tag{4}$$

The optimization problem (4) can be interpreted pictorially as Figure 3.

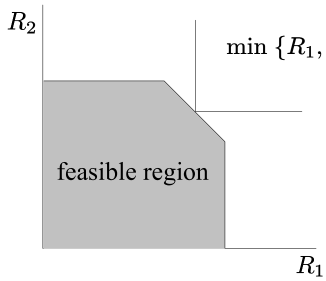

Figure 3: Graphical interpretation of the maximizing the worst-user's capacity problem for a two-user scalar channel

Thus, no matter how the feasible region is shaped, the solution of the optimal rate pair (R_1, R_2) can be found at the intersection of the boundary of the capacity region with the 45 degree line, as shown in Figure 4.

When the products of the square of channel gain and the given power are comparable between the two users, the capacity region will look like (a) in the Figure 4. If they are not, the region will shape like either (b) or (c) depending on the product. To reach the optimal point with respect to maximizing the worst user's capacity for (b), a transmitter is required to encode $((2^{nR}, 2^{nR}), n)$ codes where $R = 0.5\log_2(1+h_2P_2)$ and a receiver decode user 1 code regarding user 2 as an interference and then decode user 2 code. A similar argument can be applied for (c). But for (a), time sharing is required to reach the optimal point.

We now consider the problem for multiple users. Like the two-user case, the rate vector that maximize the minimum capacity is determined where the straight line $r_1 = r_2 = \cdots = r_N$ is touching the boundary of the capacity region. It implies that all users have the same rate at this optimal point. That is, it tells us what the minimum rate is to receive information from all users with an equal rate.

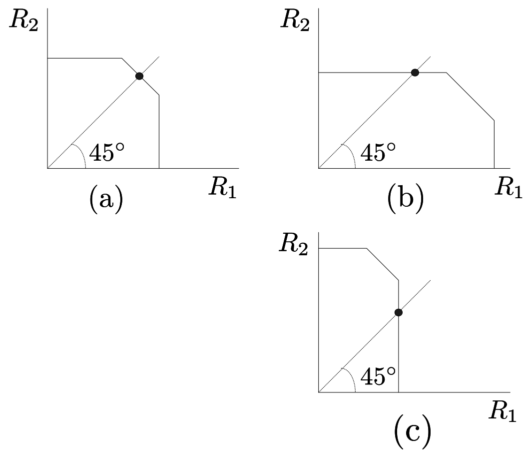

Figure 4: Optimal rate pair (R_1, R_2) examples

4 Vector Channel

4.1 Single-User Vector Channel

The single-user vector channel model can be interpreted as a parallel channel since the channel matrix has only diagonal terms. This model can be written as

$$\mathbf{Y} = \mathbf{HX} + \mathbf{Z} \tag{5}$$

where $\mathbf{X}$ is the transmitter's random vector, $\mathbf{Y}$ is the receiver's random vector, $\mathbf{H}$ is the channel gain matrix, and $\mathbf{Z}$ is an additive Gaussian noise with $\mathcal{N}(\mathbf{0}, \mathbf{I})$. Note that bold face letters denote matrix-vector quantities. WLOG, the channel gain matrix is assumed to be $\mathbf{H} = \mathrm{diag}(\sqrt{h_1}, \sqrt{h_2}, \cdots, \sqrt{h_K})$ and $h_1 \geq h_2 \geq \cdots \geq h_K \geq 0$. And the transmitter has power constraint of $\mathrm{tr}(E[\mathbf{XX}^T]) \leq P$.

The optimal allocation strategy to maximize the channel capacity is given by the water-filling algorithm [10]. Let $\mathbf{Q} = E[\mathbf{XX}^T]$, then the optimal $\mathbf{Q}$ is

$$\left[\mathrm{diag}(\mathbf{Q})\right]_{ii} = \left(\nu - \frac{1}{h_i}\right)^+ \tag{6}$$

where ν is chosen so that $\sum (\nu - 1/h_i)^+ = P$. Note that $(a)^+ := \max\{0, a\}$. When the covariance matrix of $\mathbf{Z}$ is not the identity matrix, whitening by multiplying $\mathbf{S}_Z^{-1/2}$ is required before applying the water-filling. Note that $\mathbf{S}_Z$ is the covariance matrix of $\mathbf{Z}$ and it is a diagonal matrix since there is no interference between the subchannels.

4.2 Multi-User Vector Channel

Unlike the single-user case, we can strategically allocate resources either to maximize the system performance or to minimize the cost. This is due to the multi-user diversity. That is, a deep faded subchannel for a user may appear to be good for some other user.

We now consider the problem of maximizing the worst-user's rate for a two-user case first. The model is given as

$$\mathbf{Y} = \mathbf{H}_1\mathbf{X}_1 + \mathbf{H}_2\mathbf{X}_2 + \mathbf{Z} \tag{7}$$

where $\mathbf{H}_1$ and $\mathbf{H}_2$ are channel gains for the two users and $\mathbf{Z}$ is an additive noise with $\mathcal{N}(\mathbf{0}, \mathbf{I})$. Note that the channel gain matrices are diagonal and that there are input power constraints: $\mathrm{tr}(E[\mathbf{X}_1\mathbf{X}_1^T]) \leq P_1$ and $\mathrm{tr}(E[\mathbf{X}_2\mathbf{X}_2^T]) \leq P_2$.

Similar to the multi-user scalar channel case, the capacity region is given as

$$\begin{aligned} R_1 &\leq I(\mathbf{X}_1; \mathbf{Y}|\mathbf{X}_2) \\ R_2 &\leq I(\mathbf{X}_2; \mathbf{Y}|\mathbf{X}_1) \\ R_1 + R_2 &\leq I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}) \end{aligned} \tag{8}$$

and the mutual information is bounded as follows for the Gaussian channel:

$$\begin{aligned} I(\mathbf{X}_1; \mathbf{Y}|\mathbf{X}_2) &\leq \frac{1}{2}\log\det\left(\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^T + \mathbf{I}\right) \\ I(\mathbf{X}_2; \mathbf{Y}|\mathbf{X}_1) &\leq \frac{1}{2}\log\det\left(\mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^T + \mathbf{I}\right) \\ I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}) &\leq \frac{1}{2}\log\det\left(\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^T + \mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^T + \mathbf{I}\right) \end{aligned} \tag{9}$$

where $\mathbf{Q}_1 = E[\mathbf{X}_1\mathbf{X}_1^T]$ and $\mathbf{Q}_2 = E[\mathbf{X}_2\mathbf{X}_2^T]$. Figure 5 illustrates (9) for two channel cases.

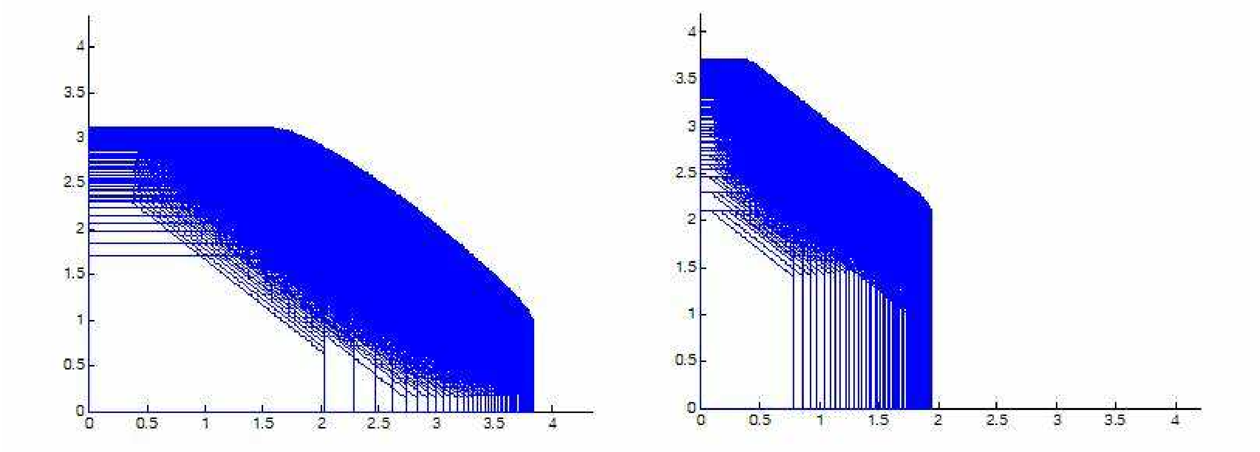

Figure 5: Capacity region examples for two-user vector channels

Tse and Hanly [7] showed that the capacity region is a polymatroid and can be characterized by Figure 6.

Note that part of the boundary surface is curved. When the 45 degree line intersects the non-curved part of the boundary, the optimization of maximizing the worst-user's rate can be solved easily by superposition coding and successive decoding. So I will focus on the problem when it intersects the curved part.

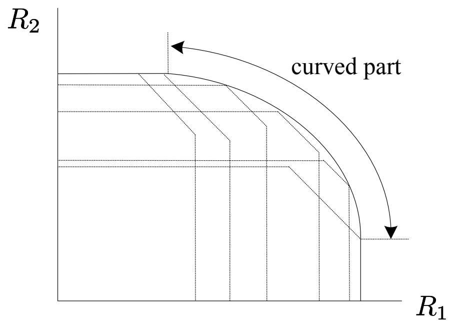

Figure 6: Characteristics of two-user capacity region

When we have the characteristics of the surface of the curved part, we can get the slope at the optimal point as in Figure 7.

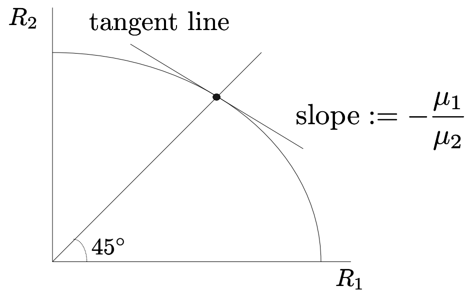

Figure 7: The slope at the optimal point

Then the problem can be re-casted into a problem which has a solution already. That is, the sum-rate capacity [8]

$$\begin{aligned} \text{maximize} \quad & \mu_1 R_1 + \mu_2 R_2 \\ \text{subject to} \quad & \text{tr}\{\mathbf{Q}_1\} \le P_1, \ \text{tr}\{\mathbf{Q}_2\} \le P_2 \end{aligned} \tag{10}$$

For the multiple-user case, it can be extended by the supporting hyperplane theorem [11]. The vector normal to the supporting hyperplane at the intersection of the boundary and the 45 degree line will be the rate vector. Achieving the boundary point can be accomplished by successive decoding.

Instead of reaching the intersection of the 45 degree line and the boundary of the capacity region explicitly, we can progressively approach the intersection in the following way: Pouring a small amount of power to user 1, user 1 will reach a certain rate R_1. Then pour power to user 2 until user 2 has the same rate with user 1. Increase user 1's power again, and do the same thing for user 2. Keep running the pouring until the given powers are exhausted. The algorithm is described as follow:

Algorithm 1:

$$
\begin{aligned}
&\mathbf{W} \leftarrow \mathbf{I} \qquad \text{noise + interference}\\
&\text{while} \quad p_1 \le P_1, \;\; p_2 \le P_2\\
&\quad p_1 \leftarrow p_1 + \Delta_1\\
&\quad \mathbf{Q}_1 \leftarrow \text{waterfill}(\mathbf{H}_1, p_1, \mathbf{W})\\
&\quad R_1 \leftarrow \frac{1}{2}\log_2 \frac{\det(\mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^T + \mathbf{W})}{\det(\mathbf{W})}\\
&\quad \mathbf{W} \leftarrow \mathbf{H}_1\mathbf{Q}_1\mathbf{H}_1^T + \mathbf{I}\\
&\quad \text{while} \quad R_1 - R_2 > tol\\
&\qquad p_2 \leftarrow p_2 + \Delta_2\\
&\qquad \mathbf{Q}_2 \leftarrow \text{waterfill}(\mathbf{H}_2, p_2, \mathbf{W})\\
&\qquad R_2 \leftarrow \frac{1}{2}\log_2 \frac{\det(\mathbf{H}_2\mathbf{Q}_2\mathbf{H}_2^T + \mathbf{W})}{\det(\mathbf{W})}\\
&\quad \text{end}\\
&\text{end}
\end{aligned}
$$

As a numerical example, when $\mathbf{H}_1 = \text{diag}(2.0918, 1.2608)$, $\mathbf{H}_2 = \text{diag}(1.1688, 1.839)$, $\text{tr}(\mathbf{Q}_1) \le 7$, and $\text{tr}(\mathbf{Q}_2) \le 10$, two user's rates reached 2.4923 and 2.4913, respectively, as shown in Figure 8. Note that the point determined by this algorithm is very close to the boundary. Although the algorithm seems to work, I have yet to determine whether the algorithm will reach the exact optimal point.

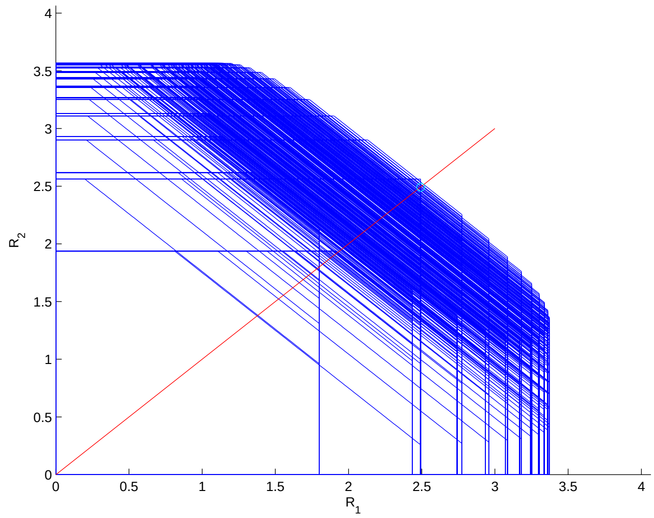

Figure 8: Example of the algorithm 1

5 Conclusion

This project has dealt with the problem of maximizing the worst-user's rate. The optimal rate was determined when all users have the same maximal rate and it can be achieved by a successive decoding for vector channel cases. Though the optimal solution may not have been determined in this project, I have showed an algorithm to close to the optimal rate with a geometrical interpretation. When the boundary surface characteristic is known, it is also pointed out that the optimization problem can be re-cast into a sum-rate capacity optimization problem.

The remaining tasks include analyzing the characteristics of the curved part of the boundary surface of the capacity region, finding an algorithm how to get to the optimal point which is given at the intersection of the boundary surface and the 45 degree line, and the extension to the multi-user cases.

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