Lectures 7-8

- Entropy of Gaussian RV's
- AWGN Channels
- Parallel AWGN Channels
- Fading Channels

Entropy of Gaussian RV

- PDF of multi-variate Gaussian RV

$$X \sim N(\vec{\mu}, K) \quad E[X] = \vec{\mu}, \quad E[X^T X] = K$$

$$f(\vec{x}) = \frac{1}{(2\pi)^{n/2} |K|^{1/2}} \exp\left(-\frac{1}{2} (\vec{x} - \vec{\mu}) K^{-1} (\vec{x} - \vec{\mu})^T \right)$$

- Entropy

$$H(X) = \frac{1}{2} \log_2 (2\pi e)^n |K| \text{ bits}$$

Entropy Maximization

- Thm: For any X with $E[XX^T]=K$, entropy bounded by Gaussian RV with same covariance:

$$H(X) \leq \frac{1}{2}\log_2(2\pi e)^n |K|$$

- Implication: Gaussians maximize entropy for fixed variance, second moments

AWGN Channel

- $Y_i = X_i + Z_i$
  - Z_i iid N(0,N), independent of X_i
  - i = time index
  - Power constraint on each codeword $x^n(w)$: 1/n $||x^n(w)||$^2 <= P for all w

- Thm:

$$C = \max_{X : E[X^2] \leq P} I(X;Y) = \frac{1}{2}\log\left(1+\frac{P}{N}\right)$$

- Achievability: Random coding & AEP
- Converse: Fano's, Data-Processing, Concavity of log

Band-Limited Channels

- $Y(t) = (X(t)+Z(t)) * h(t)$
  - $h(t)$ is low pass filter $(-W, W)$
  - $Z(t)$ white Gaussian noise, PSD = $N_0/2$
- Thm: $C = W \log(1 + P/(N_0 W))$
- Proof: Uses sampling theorem

Parallel AWGN Channels

- $Y_j = X_j + Z_j \quad j=1,\ldots,K$
  - Z_j iid $N(0,N_j)$
  - Total power constraint

$$C = \max_{X_1,\ldots,X_K : E[\sum_i X_i^2] \le P} I(X^K;Y^K)$$

- Capacity achieving strategy: Choose $X_j \sim N(0,P_j)$ (independent) with $P_j = [c - N_j]^+$, c chosen such that power constraint satisfied
- Proof: Lagrangian method

Fading Channels

- $Y_i = h_i X_i + Z_i$
  - Z_i iid N(0,N), independent of X_i
  - i = time index
  - h fading distribution (stat & ergodic process)

- Perfect CSI at TX & RX: (identical to parallel channels)

$$C = \max_{P(h):E[P(h)]\le P} E\left[\frac{1}{2}\log\left(1+hP(h)\right)\right]$$

- Perfect CSI at RX, no CSI at TX:

$$C = E\left[\frac{1}{2}\log\left(1+hP\right)\right]$$