EE 8510 Thursday, Feb. 24, 2005
Advanced Topics in Communications Prof. N. Jindal

Homework Set 5
Due: Thursday, March 3, 2005

1. *Waterfilling at High SNR.* Consider K parallel AWGN channels with noise power $N_1, \ldots, N_K$ on the K channels. We showed in class that the capacity of this channel is achieved by independent Gaussian random variables on each channel:

$$\mathcal{C}(P) = \max_{P_i : \sum_{i=1}^{K} P_i \le P} \sum_{i=1}^{K} \frac{1}{2} \log\left(1 + \frac{P_i}{N_i}\right),$$

and the optimal values of $P_1, \ldots, P_K$ are chosen according to the waterfilling procedure. In this problem we compare the capacity to the rates achievable by allocating equal power to each of the K channels, i.e. $P_i = \frac{P}{K}$:

$$\mathcal{C}_{equal}(P) = \sum_{i=1}^{K} \frac{1}{2} \log\left(1 + \frac{P}{KN_i}\right).$$

   (a) Show that

$$\lim_{P \to \infty} \left(\mathcal{C}(P) - \mathcal{C}_{equal}(P)\right) = 0.$$

   This implies that equal power allocation is optimal at asymptotically high SNR.

   (b) Use part(a) to argue that the capacity-achieving covariance matrix for a MIMO channel with fixed channel matrix $\mathbf{H}$ (assumed to be full rank) and $N_t \le N_r$ converges to the identity matrix. A rigorous proof is not required.

   (c) Can the same statement be made for the optimal covariance matrix when $N_t > N_r$? Provide a simple counterexample of a full rank channel matrix $\mathbf{H}$ such that the statement in (b) is not true.

2. *Cooperative multiple-access channel.* Consider a cooperative multiple-access channel where X_1 and X_2 each have access to *both* indices $W_1 \in \{1, \ldots, 2^{nR_1}\}$ and $W_2 \in \{1, \ldots, 2^{nR_2}\}$. Thus the codewords $X_1^n(W_1, W_2)$ and $X_2^n(W_1, W_2)$ depend on both indices.

   (a) Find the capacity region.

   (b) Evaluate this region for the binary erasure multiple-access channel $Y = X_1 + X_2,\ X_i \in \{0, 1\}$. Compare to the non-cooperative capacity region.

   (Cover & Thomas 14.1)

3. Find the capacity region for each of the following multiple-access channels:

   (a) Additive modulo-2 multiple access channel. $X_1 \in \{0,1\}$, $X_2 \in \{0,1\}$, $Y = X_1 \oplus X_2$.

   (b) Multiplicative multiple-access channel. $X_1 \in \{-1,1\}$, $X_2 \in \{-1,1\}$, $Y = X_1 \cdot X_2$.

   (Cover & Thomas 14.2)

4. *Unusual multiple access channel.* Consider the following multiple access channel: $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y} = \{0,1\}$. If $(X_1, X_2) = (0,0)$ then $Y = 0$. If $(X_1, X_2) = (0,1)$ then $Y = 1$. If $(X_1, X_2) = (1,0)$ then $Y = 1$. If $(X_1, X_2) = (1,1)$ then $Y = 0$ with probability $1/2$ and $Y = 1$ with probability $1/2$.

   (a) Show that the rates pairs $(0,1)$ and $(1,0)$ are achievable.

   (b) Show that for any non-degenerate distribution $p(x_1)p(x_2)$, we have $I(X_1, X_2; Y) < 1$.

   (c) Argue that there are points in the capacity region of this multiple access channel that can only be achieved by timesharing, i.e. there exist achievable rate pairs (R_1, R_2) which lie in the capacity region of the channel but not in the region defined by

   $$R_1 \leq I(X_1; Y|X_2), \;\; R_2 \leq I(X_2; Y|X_1), R_1 + R_2 \leq I(X_1, X_2; Y)$$

   for any product distribution $p(x_1)p(x_2)$. Hence the operation of convexification strictly enlarges the capacity region.

   (Cover & Thomas 14.6)

5. *Successive Cancellation.* Consider a discrete memoryless MAC $(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$. To achieve a corner point of a set $\mathcal{R}(X_1, X_2)$, e.g. $R_1 = I(X_1; Y|X_2) - \epsilon, R_2 = I(X_2; Y) - \epsilon$ for any $\epsilon > 0$, use random coding and the following two-step decoding scheme: the receiver first declares that $\hat{w}_2$ is sent if it is the unique message such that $(x_2^n(\hat{w}_2), y^n) \in A_\epsilon^{(n)}$, otherwise, an error is declared. If such a $\hat{w}_2$ is found, the receiver declares that $\hat{w}_1$ was sent if it is the unique message such that $(x_1^n(\hat{w}_1), x_2^n(\hat{w}_2), y^n) \in A_\epsilon^{(n)}$, otherwise an error is declared. Provide a detailed analysis of the error probability to show that the corner point is achievable. (A. El Gamal)

6. *Alternative Error Criterion.* In class we derived the capacity region with an average probability of error criterion, where we said an error occurred if our estimate of *either* message was incorrect, i.e. $P_e^{(n)} = Pr\{(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)\}$. This seems overly stringent. In this problem, we consider a separate probability of error for each user, i.e. $P_{e,1}^{(n)} = Pr\{\hat{W}_1 \neq W_1\}$ and $P_{e,2}^{(n)} = Pr\{\hat{W}_2 \neq W_2\}$, and consider a rate pair to be achievable if $P_{e,1}^{(n)}$ and $P_{e,2}^{(n)}$ can both be driven to zero. Show that this error criterion yields the same capacity region.

7. *Capacity for maximum vs. average probability of error.* We first proved the capacity of a DMC assuming average probability of error. We also showed that the capacity is the same if we consider maximum probability of error by considering a codebook with average probability of error ϵ, and using only the best half of the codewords which are guaranteed to each have probability of error less than 2ϵ by Markov's inequality. Such argument cannot be used to show that the capacity of an arbitrary DM-MAC with maximum probability of error is the same as that with average probability of error.

   (a) Argue that simply discarding half of the codeword pairs with the highest probability of error does not in general work.

   (b) How about throwing out the worst half of each sender's codewords? Show that this does not work either. (Hint: Provide a simple example of a set of probabilities $p_{ij} \in [0, 1]$, $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$, with $\frac{1}{n^2} \sum_{i,j} p_{ij} \leq \epsilon$, for some $0 < \epsilon < \frac{1}{4}$, such that there are no subsets $\mathcal{N}_i \subset \{1, 2, \ldots, n\}$ with cardinalities $|\mathcal{N}_i| \geq \frac{n}{2}$, for $i = 1, 2$, such that $p_{ij} \leq 4\epsilon$ for all $(i, j) \in \mathcal{N}_1 \times \mathcal{N}_2$.)

   Note: A much stronger statement in fact holds. Dueck in a 1978 paper provided an example of a MAC for which the capacity region with maximum probability of error is strictly smaller than that with average probability of error. This provides yet another example where a result from single-user information theory does not necessarily carry over to the multiple user case. (A. El Gamal)