EE 8510 — Tuesday, Feb. 15, 2005
Advanced Topics in Communications — Prof. N. Jindal

Homework Set 4
Due: Tuesday, Feb. 22, 2005

1. *Channel Reciprocity.* In this problem we consider two reciprocal MIMO systems, one with channel $\mathbf{H}$ and the other with channel matrix $\mathbf{H}^\dagger$:
   (a) Show that if $\mathbf{H}$ is fixed (i.e. there is perfect TX and RX CSI), then the capacity of the channels described by matrices $\mathbf{H}$ and $\mathbf{H}^\dagger$ are the same.
   (b) Show that if $\mathbf{H}$ is fading and there is only RX CSI, then the capacities of $\mathbf{H}$ and $\mathbf{H}^\dagger$ are not necessarily the same. Provide a simple example of this.

2. *Colored Noise.* Consider a standard MIMO channel:
$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$$
   where $\mathbf{H} \in \mathcal{C}^{N\times N}$ is fixed, but the covariance of the noise is given by $E[\mathbf{z}\mathbf{z}^\dagger] = \mathbf{\Sigma}_z \neq \mathbf{I}$. Calculate the capacity of this channel, subject to a power constraint P on the input. For simplicity, assume that the inverse of $\mathbf{\Sigma}_z$ exists.

3. *Matrix Identities.* A matrix $\mathbf{B}$ is Hermitian if $\mathbf{B} = \mathbf{B}^\dagger$. A Hermitian matrix $\mathbf{A}$ is positive semi-definite iff $\mathbf{x}^\dagger\mathbf{A}\mathbf{x} \geq 0$ for all vectors $\mathbf{x}$, and is positive definite iff $\mathbf{x}^\dagger\mathbf{A}\mathbf{x} > 0$ for all vectors $\mathbf{x} \neq \mathbf{0}$. Consider an arbitrary $N_r \times N_t$ matrix $\mathbf{H}$.
   (a) Show that the matrices $\mathbf{H}\mathbf{H}^\dagger$ and $\mathbf{H}^\dagger\mathbf{H}$ are each Hermitian.
   (b) Show that the matrices $\mathbf{H}\mathbf{H}^\dagger$ and $\mathbf{H}^\dagger\mathbf{H}$ are positive semi-definite, and that the matrix $\mathbf{I} + \mathbf{H}^\dagger\mathbf{H}$ is positive definite.
   (c) How are the eigenvalues and eigenvectors of $\mathbf{H}^\dagger\mathbf{H}$ related to the eigenvalues and eigenvectors of $\mathbf{I} + \mathbf{H}^\dagger\mathbf{H}$?
   (d) Prove that if a Hermitian matrix is positive definite, then its columns are linearly independent.

4. *Asymptotic Capacity.* Consider a MIMO channel with iid Rayleigh fading. In this problem you will use the fact that if $\mathbf{h}$ is an $N \times 1$ vector with iid circularly symmetric complex Gaussian components, then $||\mathbf{h}||^2$ is chi-square distributed with $2N$ degrees of freedom. Thus, $||\mathbf{h}||^2 \to N$ (roughly) as N goes to infinity.

   (a) Calculate the limit of the capacity when only the RX has CSI, with $N_t = 1$ and $N_r \to \infty$.

   (b) Calculate the limit of the capacity when only the RX has CSI, with $N_r = 1$ and $N_t \to \infty$.

   (c) Why do these results differ so drastically?

   (d) If the TX also has CSI, would the results be the same?

5. *Unitary transformation of a Gaussian.* Assume that $\mathbf{z} \in \mathcal{C}^n$ is complex circularly symmetric Gaussian with $E[\mathbf{z}] = 0$ and $E[\mathbf{z}\mathbf{z}^\dagger] = \mathbf{I}$. Show that $\mathbf{U}\mathbf{z}$ has the same distribution as $\mathbf{z}$ for any unitary matrix $\mathbf{U}$ (i.e. $\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$).