EE 8510 Tuesday, Feb. 8, 2005
Advanced Topics in Communications Prof. N. Jindal

Homework Set 3

Due: Tuesday, Feb. 15, 2005

1. *Multi-variate Gaussian random variable.* Show that if $\mathbf{x} \in \mathcal{R}^n$ is normally distributed with $E[\mathbf{x}] = \mathbf{0}$ and $E[\mathbf{x}\mathbf{x}^T] = \mathbf{K}$, then the entropy of X is given by:
$$h(\mathbf{x}) = \frac{1}{2} \log_2 (2\pi e)^n |\mathbf{K}| \text{ bits.}$$

2. *Entropy maximization.* Show that if the multi-variate random vector $\mathbf{x}$ ($\in \mathcal{R}^n$) has zero mean and covariance $\mathbf{K}$, i.e. $E[[\mathbf{x}\mathbf{x}^T] = \mathbf{K}$, then the entropy of $\mathbf{x}$ is upper bounded by the entropy of a normally distributed random variable with the same covariance:
$$h(\mathbf{x}) \leq \frac{1}{2} \log(2\pi e)^n |\mathbf{K}|$$
with equality iff $\mathbf{x}$ is Gaussian.

3. *A mutual information game.* Consider a standard additive channel: $Y = X + Z$, with the following constraints:
$$E[X] = 0, \quad E[X^2] = P, \quad E[Z] = 0, \quad E[Z^2] = N$$
and assume that X and Z are independent. The mutual information is given by $I(X; X + Z)$. Now for the game. The noise player chooses a distribution on Z to *minimize* $I(X; X+Z)$, while the signal player chooses a distribution on X to *maximize* $I(X; X + Z)$. Let $X^* \sim N(0, P)$ and $Z^* \sim N(0, N)$.
   (a) Show $I(X; X + Z^*) \leq I(X^*; X^* + Z^*)$ for any distribution on X, with equality iff $X \sim N(0, P)$. Thus, a Gaussian input maximizes the mutual information in an additive Gaussian noise channel.
   (b) Show $I(X^*; X^* + Z) \geq I(X^*; X^* + Z^*)$ for any distribution on Z, with equality iff $Z \sim N(0, N)$. Thus, Gaussian noise minimizes the mutual information (i.e. is the worst-case noise) if the input is Gaussian. In order to prove this, you will need to make use of the entropy power inequality, which states that $2^{2h(X+Z)} \geq 2^{2h(X)} + 2^{2h(Z)}$ if X and Z are independent.

(c) In parts (b) and (c) we showed

$$I(X; X + Z^*) \le I(X^*; X^* + Z^*) \le I(X^*; X^* + Z).$$

Show that this implies

$$\min_Z \max_X I(X; X + Z) = \max_X \min_Z I(X; X + Z),$$

i.e. that there is a saddlepoint in the mutual information game achieved by $X^* \sim N(0, P)$ and $Z^* \sim N(0, N)$. At this saddlepoint, neither the signal player (X) nor the noise player (Z) has any incentive to move.
Hint: Show that $\min_a \max_b f(a, b) \ge \max_b \min_a f(a, b)$ for any function $f(a, b)$, and then show inequality in the other direction by using parts (b) and (c).

(d) In part (b) we proved that Gaussian noise is the worst case noise when the input is Gaussian. However, Gaussian noise is not necessarily the worst case noise when the input is not Gaussian. In this part, you must provide an example of distributions (continuous or discrete) of X and Z for which $I(X; X + Z) < I(X; X + Z^*)$, i.e. an input distribution for which Gaussian noise is not the worst-case. You should be able to find a distribution on X and Z such that $I(X; X + Z)$ is easy to evaluate, but you may need to use MATLAB to calculate $I(X; X + Z^*)$.

(Cover & Thomas 10.1)

4. *A channel with two independent looks at X.* Let Y_1 and Y_2 be conditionally independent and conditionally identically distributed given X.

   (a) Show $I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2)$.

   (b) Conclude that the capacity of the channel from X to (Y_1, Y_2) is less than twice the capacity of the single-look channel from X to Y_1.

   (Cover & Thomas 10.2)

5. *The two-look Gaussian channel.* Consider the ordinary Gaussian channel with two correlated looks at X, i.e. $Y = (Y_1, Y_2)$ where

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$

with power constraint P on X, and $(Z_1, Z_2) \sim \mathcal{N}(\mathbf{0}, \mathbf{K})$ where

$$\mathbf{K} = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}$$

(a) Find the capacity C for $\rho = 0$, $\rho = 1$, and $\rho = -1$.

(b) Show that the capacity of the channel $X \to (Y_1, Y_2)$ and $X \to Y_1 + Y_2$ are the same for all values of ρ.

(Cover & Thomas 10.3)