EE 5581 Prof. Jindal
Information Theory & Coding October 14, 2005

Midterm 1

You have 50 minutes to complete this exam. You must show your work to receive credit.

1. **Drawing Without Replacement** (25 pts) - 2 parts

   Consider an urn with n black balls and n red balls (with $n \geq 1$). Let X_1 and X_2 denote two balls randomly drawn *without* replacement.

   (a) Compute $H(X_2|X_1)$. (15 pts)

   (b) Compute $I(X_1; X_2)$. (10 pts)

2. **Random Processes** (30 pts) - 2 parts

   Let X_1, X_2 be i.i.d. random variables taking values in $\{0, 1\}$, with $\Pr(X_1 = 1) = \Pr(X_2 = 1) = \frac{1}{2}$. For $n \geq 3$, define X_n as follows:

   $$X_n = \begin{cases} 0, & \text{if } X_{n-1} \neq X_{n-2} \\ 1, & \text{if } X_{n-1} = X_{n-2} \end{cases}$$

   This process is easily shown to be stationary, which implies $P(X_n = 1) = \frac{1}{2}$ for all n.

   (a) Compute $H(X_n|X_1, X_2)$ for $n \geq 3$. (15 pts)

   (b) What is the entropy rate of this random process? (15 pts)

3. **Source Coding** (45 pts) - 3 parts

   (a) Compute a binary Huffman code for a source with distribution $\mathbf{p} = (\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{5}{15})$. (15 pts)

   (b) Can the following codes be optimal prefix-free codes for any probability assignment? Briefly justify each answer. (15 pts)

      i. $\{0, 10, 11, 01\}$

      ii. $\{0, 1\}$

      iii. $\{00, 01, 101, 110, 111\}$

   (c) Consider the following source X:

   $$X = \begin{cases} 1, & \text{with probability } 0.4 \\ 2, & \text{with probability } p \\ 3, & \text{with probability } (0.6 - p) \end{cases}$$

   and the following binary code:

   $$C(x) = \begin{cases} 0 & \text{if x} = 1 \\ 10 & \text{if x} = 2 \\ 11 & \text{if x} = 3 \end{cases}$$

   For what values of p is $C(x)$ an optimal (i.e., shortest expected length) prefix-free code for source X? (15 pts)