The deterministic simple harmonic oscillator is governed by the equation

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = f(t)$$

or equivalently

$$\frac{dv}{dt} = -\omega^2 x - \gamma \dot{x} + f(t)$$

$$= -\omega^2 x - \gamma v + f(t)$$

$$\Rightarrow \quad dv = -\omega^2 x\,dt - \gamma v\,dt + f(t)\,dt$$

and

$$\frac{dx}{dt} = v$$

$$\Rightarrow \quad dx = v\,dt$$

Assuming that the forcing is being caused by the surrounding molecules and the damping is also due to the surrounding molecules. We can model the stochastic forcing due to surrounding molecules by

$$f(t)\,dt = \sqrt{\beta^2 dt}\; N_t^{t+dt}(0,1)$$

and thus,

$$dV = -\omega^2 x - \gamma V + \sqrt{\beta^2 dt}\; N_t^{t+dt}(0,1)$$

and $dx = V\,dt$.

Clearly as in earlier cases; assuming that $x(0)$ and $V(0)$ are deterministic we have that

$$V(dt) = -\omega^2 x(0) - \gamma V(0) + \sqrt{\beta^2 dt}\; N_0^{\cancel{t+}dt}(0,1) + V(0)$$
$$\in \mathrm{Span}\{N_0^{\cancel{t+}dt}(0,1)\}$$

$$x(dt) = V(0)\,dt + x(0) \in$$

$$V(2dt) = -\omega^2 x(dt) - \gamma V(dt) + \sqrt{\beta^2 dt}\; N_{dt}^{2dt}(0,1) + V(dt)$$

Normality

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$\therefore\ v(2dt) \in \mathrm{Span}\{N_0^{dt}(0,1),\ N_{dt}^{2dt}(0,1)\}$

$|||^{ly}$, $\quad x(2dt) \in \mathrm{Span}\{N_{0}^{dt}(0,1)\}$.

and in general

$$v(ndt) \in \mathrm{Span}\{N_0^{dt}(0,1), \dots N_{(n-1)dt}^{ndt}(0,1)\}$$

$$x(ndt) \in \mathrm{Span}\{N_0^{dt}(0,1), \dots N_{(n-2)dt}^{(n-1)dt}(0,1)\}$$

Letting $ndt = t$

$$\Rightarrow \quad v(t) \in \mathrm{Span}\{N_0^{dt}(0,1), \dots N_{t-dt}^{t}(0,1)\}$$

$$x(t) \in \mathrm{Span}\{N_0^{dt}(0,1), \dots N_{t-2dt}^{t-dt}(0,1)\}.$$

Thus, both $v(t)$ and $x(t)$ are linear combinations of the same set of independent normals and therefore jointly Gaussian. Therefore their joint pdf is completely determined by

Jointly Gaussanity of X and V

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$\langle X(t) \rangle$; $\langle V(t) \rangle$; $\langle X^2(t) \rangle$; $\langle V^2(t) \rangle$
and $\mathrm{Cov}\{X(t), V(t)\}$.

We will determine these now.

Mean of X and V

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Note that

$$V(t+dt) - V(t) = -\omega^2 x(t) - \gamma V(t)dt + \sqrt{\beta^2 dt}\, N_t(0,1)$$

$$\Rightarrow \langle V(t+dt)\rangle - \langle V(t)\rangle = -\omega^2 \langle x(t)\rangle - \gamma \langle V(t)\rangle dt$$

and $$\langle x(t+dt)\rangle - \langle x(t)\rangle = \langle V(t)\rangle dt$$

$$\Rightarrow \lim_{dt\to 0} \frac{\langle V(t+dt)\rangle - \langle V(t)\rangle}{dt} = \begin{bmatrix} -\omega^2 & -\gamma \end{bmatrix} \begin{bmatrix} \langle x(t)\rangle \\ \langle V(t)\rangle \end{bmatrix}$$

$$\Rightarrow \frac{d\langle V(t)\rangle}{dt} = \begin{bmatrix} -\omega^2 & -\gamma \end{bmatrix} \begin{bmatrix} \langle x(t)\rangle \\ \langle V(t)\rangle \end{bmatrix}$$

$$\frac{d\langle x(t)\rangle}{dt} = \begin{bmatrix} 0 & +1 \end{bmatrix} \begin{bmatrix} \langle x(t)\rangle \\ \langle V(t)\rangle \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{d\langle x(t)\rangle}{dt} \\ \frac{d\langle V(t)\rangle}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix} \begin{bmatrix} \langle x(t)\rangle \\ \langle V(t)\rangle \end{bmatrix}$$

with initial conditions

$\langle x(0)\rangle = X_0$ and

$\langle V(0)\rangle = V_0$.

Mean of X and V

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Letting $A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & \gamma \end{bmatrix}$

and letting $\delta = \begin{bmatrix} \langle x(t) \rangle \\ \langle v(t) \rangle \end{bmatrix}$

we have

$$\frac{d\delta}{dt} = A\delta \quad \text{with} \quad \delta(0) = \begin{bmatrix} \langle x(0) \rangle \\ \langle v(0) \rangle \end{bmatrix}$$

and this has a solution

$$\delta(t) = e^{At}\,\delta(0)$$

where

$$e^{At} := \left(I + At + \frac{A^2 t^2}{2!} + \cdots \right)$$

$$= \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$$

Mean of X and V

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One can solve for e^{At} in closed form

to obtain

$$\langle x(t)\rangle = e^{-\gamma t/2}\left[x_0 \cos(\omega' t) + \left(v_0 + \gamma \frac{x_0}{2}\right)\frac{\sin \omega' t}{\omega'}\right] \quad \text{---} (9.1.9)$$

and

$$\langle v(t)\rangle = e^{-\gamma t/2}\left[v_0 \cos \omega' t - \left(x_0 \omega^2 + \gamma \frac{v_0}{2}\right)\frac{\sin \omega' t}{\omega'}\right] \quad \text{----} (9.1.10)$$

where $\omega' = \sqrt{\omega^2 - \gamma^2/4}$; $x(0) = x_0$

$v(0) = v_0$.

Variance of V

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Note that

$$dV = -\omega^2 x\,dt - \gamma V\,dt + \sqrt{\beta^2 dt}\,N_t^{t+dt}(0,1)$$

and therefore

$$dv^2 = v^2(t+dt) - v^2(t)$$

$$= (v+dv)^2 - v^2 = v^2 + 2v\,dv + (dv)^2 - v^2$$

$$= 2v\,dv + (dv)^2$$

$$\therefore \quad d\langle v^2\rangle = \langle dv^2\rangle = 2\langle v\,dv\rangle + \langle (dv)^2\rangle$$

$$= 2\langle -\omega^2 x V dt - \gamma v^2 dt + \sqrt{\beta^2 dt}\, v\, N_t^{t+dt}(0,1)\rangle$$

$$+ \langle (-\omega^2 x)^2 dt^2 + (\gamma v\, dt)^2 + \left(\sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1)\right)^2$$

$$+ 2\langle (-\omega^2 x)(-\gamma v)\rangle dt^2$$

$$+ 2\langle -\omega^2 x \sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1)\rangle dt$$

$$+ 2\langle -\gamma v \sqrt{\beta^2 dt}\, N_t^{t+dt}(0,1)\rangle dt$$

$$= -2\omega^2\langle xv\rangle dt - 2\gamma\langle v^2\rangle dt + 0$$

$$+ \langle -\omega^2 x\rangle dt^2 + \langle \gamma^2 v\rangle dt^2 + \beta^2 dt$$

$$-2\omega^2\gamma\langle xv\rangle dt^2 + 0 + 0$$

Ode for var{V}

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$$\therefore \frac{d\langle v^2\rangle}{dt} = -2\omega^2 \langle x, v\rangle - 2\gamma \langle v^2\rangle + \beta^2$$

We already have

$$\frac{d\langle v\rangle^2}{dt} = 2\langle v\rangle \frac{d\langle v\rangle}{dt}$$

$$= 2\langle v\rangle \left[-\omega^2 \langle x\rangle - \gamma\langle v\rangle\right]$$

$$= -\omega^2\, 2\langle v\rangle\langle x\rangle - \gamma\langle v\rangle^2$$

$$\therefore \frac{d(\langle v^2\rangle - \langle v\rangle^2)}{dt} = -2\omega^2 \operatorname{Cov}\{x,v\} - 2\gamma \operatorname{Var}\{v\} + \beta^2$$

$$\therefore \frac{d \operatorname{Var}\{v\}}{dt} = -2\omega^2 \operatorname{Cov}\{x,v\} - 2\gamma \operatorname{Var}\{v\} + \beta^2.$$

Now.

Ode for Var{X}

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$$dx^2 = x(t+dt)^2 - x^2(t)$$

$$= (x+dx)^2 - x^2$$

$$= x^2 + (dx)^2 + 2x\,dx - x^2$$

$$= (dx)^2 + 2x\,dx$$

$$= (v\,dt)^2 + 2x\,dx$$

$$= v^2 dt^2 + 2x\,dx. = v^2 dt^2 + 2xv\,dt$$

$$\therefore \langle dx^2 \rangle = \langle v^2 dt^2 \rangle + 2\langle xv \rangle dt$$

$$\frac{d\langle x^2 \rangle}{dt} = 2\langle xv \rangle$$

$$\frac{d\langle x \rangle^2}{dt} = 2\langle x \rangle \frac{d\langle x \rangle}{dt} = 2\langle x \rangle \langle v \rangle$$

$$\therefore \frac{d(\langle x^2 \rangle - \langle x \rangle^2)}{dt} = 2(\langle xv \rangle - \langle x \rangle \langle v \rangle)$$

$$= 2\,\mathrm{Cov}(x v).$$

Ode for Cov{X,V}

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$$d(xv) = x(t+dt)\,v(t+dt) - x(t)v(t)$$

$$= (x+dx)(v+dv) - xv$$

$$= xv + xdv + vdx + dxdv - xv$$

$$= xdv + vdx + dxdv$$

$$= xdv + v^2 dt + vdv\,dt$$

$$d\langle xv\rangle = \langle dxv\rangle = \langle xdv\rangle + \langle v^2 dt\rangle + \langle vdv\,dt\rangle$$

$$= \langle x(-\omega^2 x dt - \gamma v dt + N_t^{t+dt}(0,1))\rangle$$

$$+ \langle v^2\rangle dt + \langle v(-\omega^2 x dt - \gamma v dt + N_t^{t+dt}(0,1))\rangle dt$$

$$= -\omega^2\langle x^2\rangle dt - \gamma\langle x,v\rangle dt + 0 + \langle v^2\rangle dt + O(dt^2)$$

$$\therefore \frac{d\langle xv\rangle}{dt} = -\omega^2\langle x^2\rangle - \gamma\langle x,v\rangle dt + \langle v^2\rangle dt.$$

$$\frac{d\langle x\rangle\langle v\rangle}{dt} = \langle x\rangle\frac{d\langle v\rangle}{dt} + \langle v\rangle\frac{d\langle x\rangle}{dt}$$

Solution for second order statistics

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$$= \langle x \rangle \left(-\omega^2 \langle x \rangle - \gamma \langle v \rangle\right)$$

$$+ \langle v \rangle \langle v \rangle$$

$$= -\omega^2 \langle x \rangle^2 - \gamma \langle x \rangle \langle v \rangle + \langle v \rangle^2$$

$$\Rightarrow \quad \frac{d}{dt} \mathrm{Cov}\{x, v\} = -\omega^2 \mathrm{Var}\{x\} - \gamma \mathrm{Cov}\{x, v\} + \mathrm{Var}\{v\}$$

The three differential equations are

$$\frac{d}{dt} \mathrm{Cov}\{x, v\} = -\omega^2 \mathrm{Var}\{x\} - \gamma \mathrm{Cov}\{x, v\} + \mathrm{Var}\{v\}. \quad (*)$$

$$\frac{d \mathrm{Var}\{x\}}{dt} = 2 \mathrm{Cov}\{x, v\} \quad --- \quad (**)$$

$$\frac{d \mathrm{Var}\{v\}}{dt} = -2\gamma \mathrm{Var}\{v\} - 2\omega^2 \mathrm{Cov}\{x, v\} + \beta^2 \quad (***)$$

From $(*)$, $(**)$ and $(***)$ we obtain

$$\frac{d^3 \mathrm{Var}\{x\}}{dt^3} + 3\gamma \frac{d^2 \mathrm{Var}\{x\}}{dt^2} + (4\omega^2 + 2\gamma^2) \frac{d \mathrm{Var}\{x\}}{dt} + 4\gamma\omega^2 \left(\mathrm{Var}\{x\} - \beta^2 / 2\gamma\omega^2\right) = 0$$

Solution for second order statistics

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Defining $y = \mathrm{Var}\{x\} - \dfrac{\beta^2}{2\gamma\omega^2}$

$$\frac{d^3y}{dt^3} + 3\gamma\frac{d^2y}{dt^2} + (4\omega^2 + 2\gamma^2)\frac{dy}{dt} + 4\gamma\omega^2 y = 0.$$

which can be solved to obtain.

$$\mathrm{Var}\{x\} = \frac{\beta^2}{2\gamma\omega^2} + e^{-\gamma t}\left(\frac{\beta^2}{8\gamma\omega^2\omega'^2}\right) \times \left\{-4\omega^2 + \gamma^2\cos(2\omega' t) - 2\gamma\omega'\sin(2\omega' t)\right\}$$

$$\mathrm{Cov}\{x, v\} = e^{-\gamma t}\left(\frac{\beta^2}{4\omega'^2}\right)\left[1 - \cos 2\omega' t\right]$$

$$\mathrm{Var}\{v\} = \frac{\beta^2}{2\gamma} + e^{-\gamma t}\frac{\beta^2}{8\gamma\omega'^2}\left[-4\omega^2 + \gamma^2\cos(2\omega' t) + 2\gamma\omega'\sin(2\omega' t)\right]$$

Thermal Equilibrium

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As $\gamma t \to \infty$ $\quad Var\{v\} \longrightarrow \beta^2/2\gamma$

and $\frac{1}{2} M \, Var\{v\} = \frac{1}{2} k_B T$

$$\Rightarrow \boxed{\frac{\beta^2}{2\gamma} = k_B T / M.}$$