Lagrange’s Method

Lagrangian Dynamics

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial \mathbf{T}}{\partial \dot{\mathbf{q}}_{\mathbf{i}}}\right)-\frac{\partial \mathbf{T}}{\partial \mathbf{q}_{\mathbf{i}}}+\frac{\partial \mathbf{V}}{\partial \mathbf{q}_{\mathbf{i}}}=\mathbf{F}_{\mathbf{i}}$$

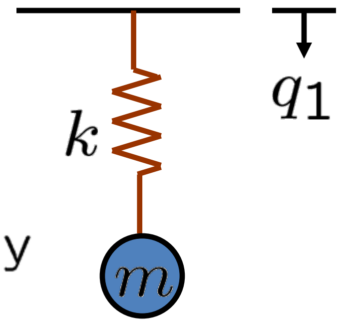
- ⋆ q_i: generalized coordinates, $1 \leq i \leq N$
- ⋆ $\dot{q}_i$: generalized velocities
- ⋆ T is the total Kinetic Energy in the system
- ⋆ V is the total Potential Energy
- ⋆ F_i: generalized non conservative forces
  - – a force for each coordinate

- need to study
  - ⋆ generalized coordinates (position, angle, charge, etc.) and constraints
  - ⋆ Kinetic Energy and Potential Energy (you already know)
  - ⋆ conservative and nonconservative Forces

- $L = T - V$ is called the *Lagrangian* of the system ... then the above equation is the same as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=F_i$$

A Trivial Example: Spring-Mass System



- ⋆ q_1: position of the mass, only 1 DOF
- ⋆ $\dot{q}_1$: velocity of the mass
- ⋆ $T = \frac{1}{2}m\dot{q}_1^2$ is the total Kinetic Energy
- ⋆ $V = \frac{1}{2}kq_1^2 - mgq_1$ is the total Potential Energy
- ⋆ $F_1 = 0$ no non conservative forces

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial \mathbf{T}}{\partial \dot{\mathbf{q}}_1}\right) - \frac{\partial \mathbf{T}}{\partial \mathbf{q}_1} + \frac{\partial \mathbf{V}}{\partial \mathbf{q}_1} = \mathbf{F}_1$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_1}\left(\frac{1}{2}m\dot{q}_1^2\right)\right) - \frac{\partial}{\partial q_1}\left(\frac{1}{2}m\dot{q}_1^2\right) + \frac{\partial}{\partial q_1}\left(\frac{1}{2}kq_1^2 - mgq_1\right) = 0$$
$$\Rightarrow \frac{d}{dt}(m\dot{q}_1) - \frac{\partial}{\partial q_1}\left(\frac{1}{2}m\dot{q}_1^2\right) + \frac{\partial}{\partial q_1}\left(\frac{1}{2}kq_1^2 - mgq_1\right) = 0$$
$$\Rightarrow m\ddot{q}_1 + 0 + kq_1 - mg = 0$$

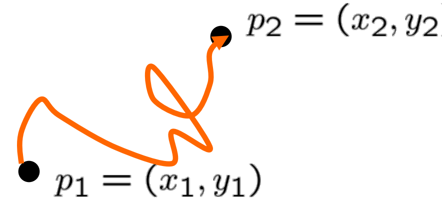
$$m\ddot{q}_1 + kq_1 = mg$$

Kinetic and Potential Energy

- Kinetic Energy: Energy by virtue of its motion:$\frac{1}{2}m(\dot{r}.\dot{r})$
  - ⋆ e.g.$\frac{1}{2}m\dot{x}^2, \frac{1}{2}J_c\dot{\theta}^2$
- Loosely speaking
  - ⋆ $V = V_{\text{elastic}} + V_{\text{gravity}}$
  - ⋆ ‘stored’ energies
    - V_{elastic}: stored in springs
      - e.g. $\frac{1}{2}kx^2, \frac{1}{2}K\theta^2$
    - V_{gravity}: stored in mass
      - mgy where y is taken w.r.t. some fixed point
      - V_{gravity} is ’extra’ P.E. from this point
- Change in energy of taking ‘it’ from point A to point B is $V_B - V_A$
  - ⋆ does not depend on the path
  - ⋆ similarly you can have Electrical Potential Energy etc.

Conservative Forces

- Conservative forces:
  - ⋆ forces derived from P.E. V: $F_i = -\frac{\partial V}{\partial q_i}$
    - – e.g. $F = -kx = -\frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right)$
    - – e.g. $F = mg = -\frac{\partial}{\partial y}(-mgy)$
  - ⋆ work done by force is independent of the path (depends only on the end points)
    - – e.g. $mg(y_2 - y_1)$ what ever the path from p_1 to p_2

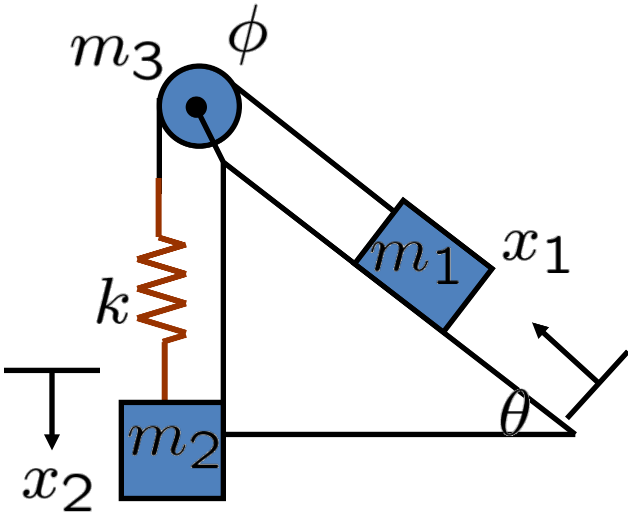


  - ⋆ mathematically: $\nabla \times F = 0$
    - – equivalently: $(\frac{\partial}{\partial q_1}\hat{e}_1 + \frac{\partial}{\partial q_2}\hat{e}_2 + \cdots) \times (F_1\hat{e}_1 + F_2\hat{e}_2 + \cdots) = 0$
    - – e.g. $F = -kx\hat{i} + mg\hat{j} \Leftrightarrow F = -kq_1\hat{e}_1 + mg\hat{e}_2$

$$
\begin{aligned}
&(\tfrac{\partial}{\partial q_1}\hat{e}_1 + \tfrac{\partial}{\partial q_2}\hat{e}_2) \times (F_1\hat{e}_1 + F_2\hat{e}_2) \\
&= (\tfrac{\partial}{\partial x}\hat{i} + \tfrac{\partial}{\partial y}\hat{j}) \times (-kx\hat{i} + mg\hat{j}) \\
&= \left(\tfrac{\partial}{\partial x}(mg) - \tfrac{\partial}{\partial y}(kx)\right)\hat{k} = 0
\end{aligned}
$$

Example

- $$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial \mathbf{T}}{\partial \dot{\mathbf{q}}_\mathbf{i}}\right) - \frac{\partial \mathbf{T}}{\partial \mathbf{q}_\mathbf{i}} + \frac{\partial \mathbf{V}}{\partial \mathbf{q}_\mathbf{i}} = \mathbf{F}_\mathbf{i}$$
  - ⋆ $q_1 = x_1,\ q_2 = x_2,\ x_1 = R\phi \Rightarrow \dot{x}_1 = R\dot{\phi}$
  - ⋆ $V = m_1 g x_1 \sin\theta - m_2 g x_2 + \frac{1}{2}k(x_1 - x_2)^2$
  - ⋆ $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}\left(\frac{1}{2}m_3 R^2\right)\left(\frac{\dot{x}_1}{R}\right)^2$
  - ⋆ $F_1 = 0$ and $F_2 = 0$

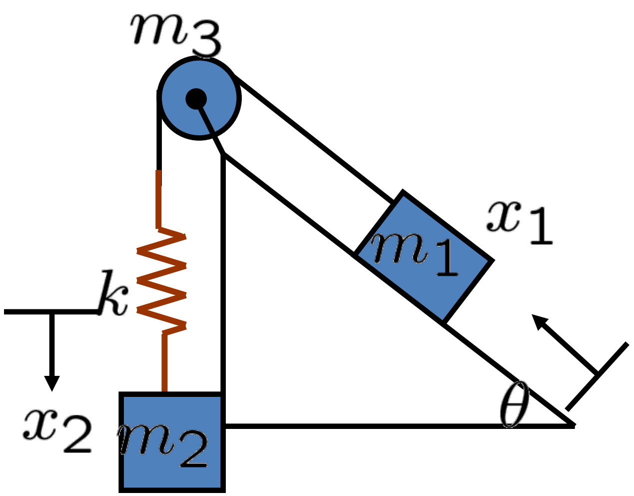


- Two Equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} = 0 \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} = 0 \qquad (2)$$

Example (cont'd.)



- Equation 1:
  - ⋆ $\frac{\partial T}{\partial \dot{x}_1} = \frac{\partial}{\partial \dot{x}_1}\left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{4}m_3\dot{x}_1^2\right)$
    $= m_1\dot{x}_1 + 0 + \frac{m_3}{2}\dot{x}_1 = (m_1 + \frac{m_3}{2})\dot{x}_1$
  - ⋆ $\frac{\partial T}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{4}m_3\dot{x}_1^2\right) = 0$
  - ⋆ $\frac{\partial V}{\partial x_1} = \frac{\partial}{\partial x_1}\left(m_1 g x_1 \sin\theta - m_2 g x_2 + \frac{1}{2}k(x_1 - x_2)^2\right)$
    $= m_1 g \sin\theta + k(x_1 - x_2)$

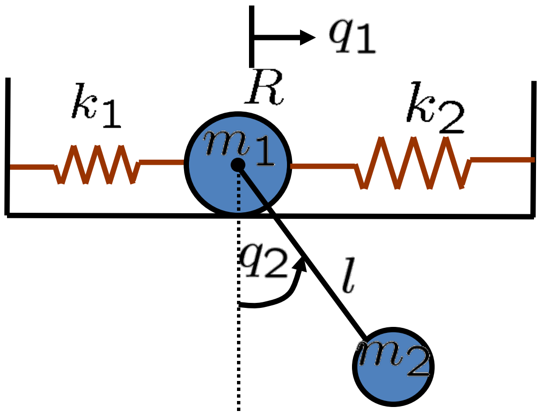
  Therefore $(m_1 + \frac{m_3}{2})\ddot{x}_1 + m_1 g \sin\theta + k(x_1 - x_2) = 0$

- Equation 2:
  - ⋆ $\frac{\partial T}{\partial \dot{x}_2} = \frac{\partial}{\partial \dot{x}_2}\left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{4}m_3\dot{x}_1^2\right) = m_2\dot{x}_2$
  - ⋆ $\frac{\partial T}{\partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{4}m_3\dot{x}_1^2\right) = 0$
  - ⋆ $\frac{\partial V}{\partial x_2} = \frac{\partial}{\partial x_2}\left(m_1 g x_1 \sin\theta - m_2 g x_2 + \frac{1}{2}k(x_1 - x_2)^2\right)$
    $= -m_2 g - k(x_1 - x_2)$

  Therefore $m_2\ddot{x}_2 - m_2 g + k(x_2 - x_1) = 0$

Example

- $$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial \mathbf{T}}{\partial \dot{\mathbf{q}}_\mathbf{i}}\right) - \frac{\partial \mathbf{T}}{\partial \mathbf{q}_\mathbf{i}} + \frac{\partial \mathbf{V}}{\partial \mathbf{q}_\mathbf{i}} = \mathbf{F}_\mathbf{i}$$



  - ⋆ $V = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2 q_1^2 - mgl\cos q_2$
  - ⋆ $T = T_{\text{disc}} + T_{\text{pendulum}}$
    - – $T_{\text{disc}} = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{2}m_1 R^2\right)\left(\frac{\dot{q}_1}{R}\right)^2 = \frac{3}{4}m_1\dot{q}_1^2$
    - – T_{pendulum}: Now $r = \begin{pmatrix} q_1 + l\sin q_2 \\ l\cos q_2 \end{pmatrix} \Rightarrow \dot{r} = \begin{pmatrix} \dot{q}_1 + l\dot{q}_2\cos q_2 \\ -l\dot{q}_2\sin q_2 \end{pmatrix}$
    - – Therefore $T_{\text{pendulum}} = \frac{1}{2}m_2|\dot{r}|^2 = \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)$

    $T = \frac{3}{4}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)$

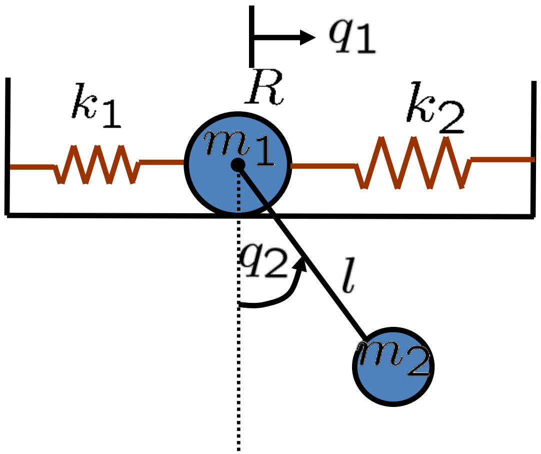
- Two Equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} = 0 \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial V}{\partial q_2} = 0 \qquad (2)$$

Example (cont'd.)

- Equation 1:
  - ⋆ $\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial}{\partial \dot{q}_1}\left(\frac{3}{4}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)\right)$
    $= \frac{3}{2}m_1\dot{q}_1 + m_2\dot{q}_1 + m_2\dot{q}_2 l\cos q_2$
    $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = (\frac{3}{2}m_1 + m_2)\ddot{q}_1 + m_2 l\cos q_2\ddot{q}_2 - m_2 l\sin q_2\dot{q}_2^2$
  - ⋆ $\frac{\partial T}{\partial q_1} = \frac{\partial}{\partial q_1}\left(\frac{3}{4}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)\right) = 0$
  - ⋆ $\frac{\partial V}{\partial q_1} = \frac{\partial}{\partial q_1}\left(\frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2 q_1^2 - m_2 gl\cos q_2\right)$
    $= (k_1 + k_2)q_1$

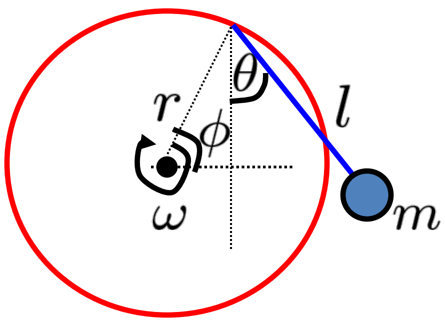


Therefore $(\frac{3}{2}m_1 + m_2)\ddot{q}_1 + m_2 l\cos q_2\ddot{q}_2 - m_2 l\sin q_2\dot{q}_2^2 + (k_1 + k_2)q_1 = 0$

- Equation 2:
  - ⋆ $\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial}{\partial \dot{q}_2}\left(\frac{3}{4}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)\right)$
    $= m_2(\dot{q}_1 l\cos q_2 + l^2\dot{q}_2)$
    $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m_2(\ddot{q}_1 l\cos q_2 - \dot{q}_1\dot{q}_2 l\sin q_2 + l^2\ddot{q}_2)$
  - ⋆ $\frac{\partial T}{\partial q_2} = \frac{\partial}{\partial q_2}\left(\frac{3}{4}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)\right) = -m_2\dot{q}_1\dot{q}_2 l\sin q_2$
  - ⋆ $\frac{\partial V}{\partial q_2} = \frac{\partial}{\partial q_2}\left(\frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2 q_1^2 - m_2 gl\cos q_2\right) = m_2 gl\sin q_2$
  - ⋆ Therefore $m_2(\ddot{q}_1 l\cos q_2 - \dot{q}_1\dot{q}_2 l\sin q_2 + l^2\ddot{q}_2) + m_2\dot{q}_1\dot{q}_2 l\sin q_2 + m_2 gl\sin q_2 = 0$

Therefore $\ddot{q}_1\cos q_2 + l\ddot{q}_2 + g\sin q_2 = 0$

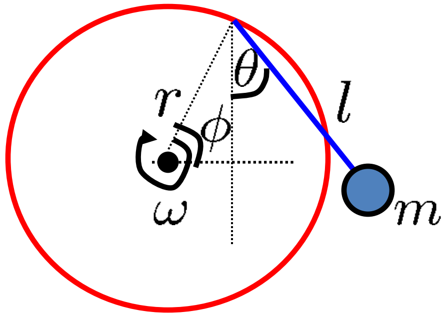
Example - with constraint



- $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = F_i$$
  - ⋆ 1 DOF: angle θ, $\dot{\phi} = \omega \Rightarrow \phi = \omega t$
    - – $p = \begin{pmatrix} r\cos\phi + l\sin\theta \\ -r\sin\phi + l\cos\theta \end{pmatrix} \Rightarrow \dot{p} = \begin{pmatrix} -r\omega\sin\phi + l\dot{\theta}\cos\theta \\ -r\omega\cos\phi - l\dot{\theta}\cos\theta \end{pmatrix}$
  - ⋆ $V = -mg(-r\sin\phi + l\cos\theta) = mg(r\sin\phi - l\cos\theta)$
  - ⋆ $T = \frac{1}{2}m|\dot{p}|^2 = \frac{1}{2}m(r^2\omega^2 + l^2\dot{\theta}^2 + 2rl\omega\dot{\theta}\sin(\theta - \omega t))$
  - ⋆ Virtual Work: $\delta W_{nc} = 0$
- Equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \quad (1)$$

Example (cont'd.)



- Equation 1:
  - ⋆ $\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} m (r^2 \omega^2 + l^2 \dot{\theta}^2 + 2rl\omega\dot{\theta} \sin(\theta - \phi)) \right)$
    $= ml^2 \dot{\theta} + mr\omega \sin(\theta - \phi)$
    $\Rightarrow \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) = ml^2 \ddot{\theta} + mr\omega(\dot{\theta} - \omega) \cos(\theta - \phi)$
  - ⋆ $\frac{\partial T}{\partial \theta} = mrl\omega\dot{\theta} \cos(\theta - \phi)$
  - ⋆ $\frac{\partial V}{\partial \theta} = \frac{\partial}{\partial \theta} (mg(r \sin\phi - l \cos\theta)) = mgl \sin\theta$

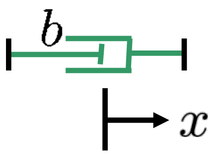
  Therefore $ml^2 \ddot{\theta} + mr\omega(\dot{\theta} - \omega) \cos(\theta - \phi) - mrl\omega\dot{\theta} \cos(\theta - \phi) + mgl \sin\theta = 0$

$$\ddot{\theta} + \frac{g}{l} \sin\theta - \frac{r\omega^2}{l} \cos(\theta - \omega t) = 0$$

Non-Conservative Forces

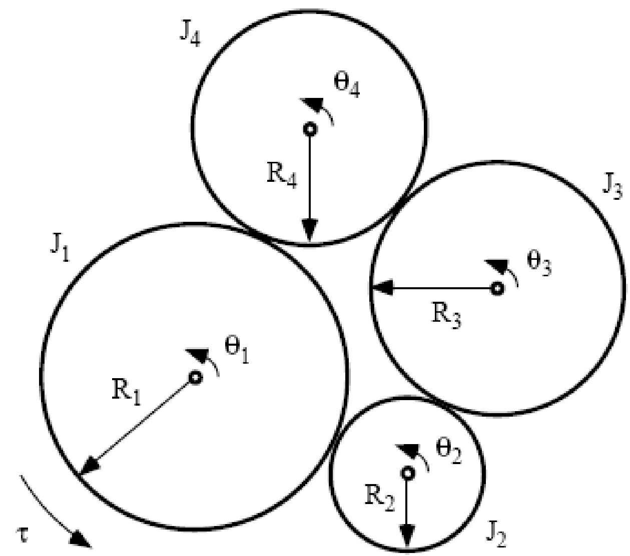
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = F_i$$

- Generalized nonconservative forces:
  - ⋆ forces not derivable from a P.E. function
    - e.g. applied forces $f(t)$
    - e.g. frictional forces, damping forces ($F = -b\dot{x}$)
  - ⋆ work done by force depends on the path (not only on the end points)
  - ⋆ they are given in terms of virtual work
    - δW_{nc} is the work done when the system coordinates are perturbed by small virtual displacements δq_i e.g. $\delta W_{nc} = -b\dot{x}\delta x$
  - ⋆ generalized force F_i is given by $F_i = \left.\frac{\delta W_{nc}}{\delta q_i}\right|_{\delta q_j = 0, j \neq i}$



Example

- $$\frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\partial \mathbf{T}}{\partial \dot{\mathbf{q}}_\mathrm{i}}\right) - \frac{\partial \mathbf{T}}{\partial \mathbf{q}_\mathrm{i}} + \frac{\partial \mathbf{V}}{\partial \mathbf{q}_\mathrm{i}} = \mathbf{F}_\mathrm{i}$$
  - ⋆ $q_1 = \theta_1, R_1\dot{\theta}_1 = -R_2\dot{\theta}_2 = R_3\dot{\theta}_3 = -R_4\dot{\theta}_4$
  - ⋆ $V = 0$
  - ⋆ $T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 + J_3\dot{\theta}_3^2 + \frac{1}{2}J_4\dot{\theta}_4^2$
    $= \frac{1}{2}\left(J_1 + \left(\frac{R_1}{R_2}\right)^2 J_2 + \left(\frac{R_1}{R_3}\right)^2 J_3 + \left(\frac{R_1}{R_4}\right)^2 J_4\right)\dot{\theta}_1^2$
  - ⋆ $\delta W_{nc} = \tau\delta\theta_1$
    $\Rightarrow F_1 = \lim_{\delta\theta_1\to 0}\frac{\delta W_{nc}}{\delta\theta_1} = \lim_{\delta\theta_1\to 0}\frac{\tau\delta\theta_1}{\delta\theta_1} = \tau$



- Equation:
  $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = F_1$$
  - ⋆ $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = \left(J_1 + \left(\frac{R_1}{R_2}\right)^2 J_2 + \left(\frac{R_1}{R_3}\right)^2 J_3 + \left(\frac{R_1}{R_4}\right)^2 J_4\right)\ddot{\theta}_1$
  - ⋆ $\frac{\partial T}{\partial \theta_1} = 0$ and $\frac{\partial V}{\partial \theta_1} = 0$
  - ⋆ Therefore $\left(J_1 + \left(\frac{R_1}{R_2}\right)^2 J_2 + \left(\frac{R_1}{R_3}\right)^2 J_3 + \left(\frac{R_1}{R_4}\right)^2 J_4\right)\ddot{\theta}_1 - 0 + 0 = \tau$

$$\Rightarrow J_{eq}\dot{\theta}_1 = \tau$$

where $J_{eq} = \left(J_1 + \left(\frac{R_1}{R_2}\right)^2 J_2 + \left(\frac{R_1}{R_3}\right)^2 J_3 + \left(\frac{R_1}{R_4}\right)^2 J_4\right)$

Example- with non conservative 'forces'

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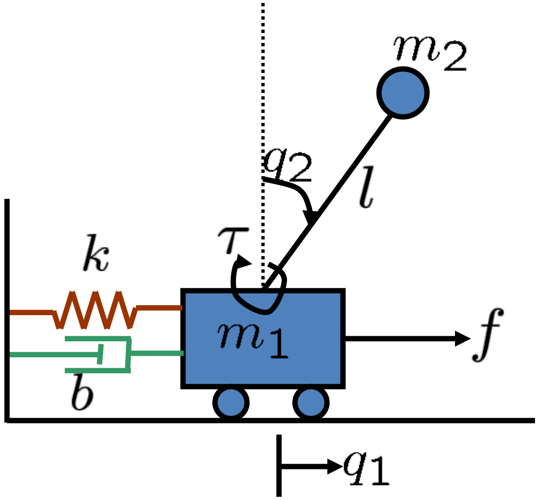
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = F_i$$



⋆ 2 DOF: position q_1 and angle q_2

⋆ $V = \frac{1}{2}kq_1^2 + m_2 gl\cos q_2$

⋆ $T = T_{\text{cart}} + T_{\text{pendulum}}$

– $T_{\text{cart}} = \frac{1}{2}m_1\dot{q}_1^2$

– T_{pendulum}: Now $r = \begin{pmatrix} q_1 + l\sin q_2 \\ l\cos q_2 \end{pmatrix} \Rightarrow \dot{r} = \begin{pmatrix} \dot{q}_1 + l\dot{q}_2\cos q_2 \\ -l\dot{q}_2\sin q_2 \end{pmatrix}$

– Therefore $T_{\text{pendulum}} = \frac{1}{2}m_2|\dot{r}|^2 = \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)$

$T = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)$

⋆ Virtual Work: $\delta W_{nc} = f\delta q_1 - b\dot{q}_1\delta q_1 + \tau\delta q_2$

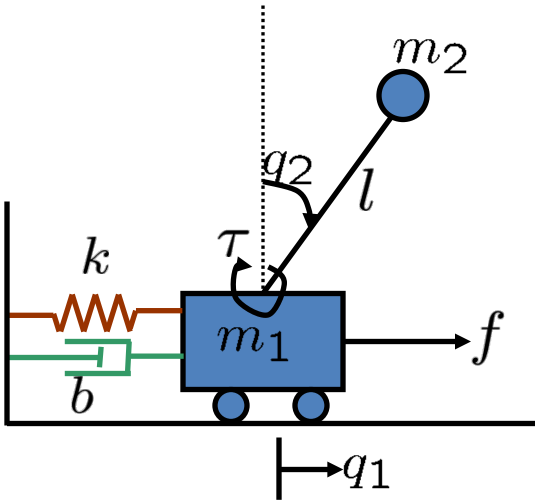
⋆ Equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} = \left.\frac{\delta W_{nc}}{\delta q_1}\right|_{q_2=0} \quad (1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial V}{\partial q_2} = \left.\frac{\delta W_{nc}}{\delta q_2}\right|_{q_1=0} \quad (2)$$

Example (cont'd.)

- Equation 1:
  - ⋆ $\frac{\partial T}{\partial \dot{q}_1} = \frac{\partial}{\partial \dot{q}_1}\left(\frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)\right)$
    $= m_1\dot{q}_1 + m_2\dot{q}_1 + m_2\dot{q}_2 l\cos q_2$
    $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = (m_1+m_2)\ddot{q}_1 + m_2 l\cos q_2\ddot{q}_2 - m_2 l\sin q_2\dot{q}_2^2$
  - ⋆ $\frac{\partial T}{\partial q_1} = 0$
  - ⋆ $\frac{\partial V}{\partial q_1} = \frac{\partial}{\partial q_1}\left(\frac{1}{2}k_1 q_1^2 + mgl\cos q_2\right) = kq_1$
  - ⋆ $F_1 = \left.\frac{\delta W_{nc}}{\delta q_1}\right|_{\delta q_2=0} = \left.\frac{f\delta q_1 - b\dot{q}_1\delta q_1 + \tau\delta q_2}{\delta q_1}\right|_{\delta q_2=0} = f - b\dot{q}_1$



Therefore $(m_1+m_2)\ddot{q}_1 + m_2 l\ddot{q}_2\cos q_2 - m_2 l\dot{q}_2^2\sin q_2 + kq_1 = f - b\dot{q}_1$

- Equation 2:
  - ⋆ $\frac{\partial T}{\partial \dot{q}_2} = \frac{\partial}{\partial \dot{q}_2}\left(m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)\right) = m_2(\dot{q}_1 l\cos q_2 + l^2\dot{q}_2)$
    $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m_2(\ddot{q}_1 l\cos q_2 - \dot{q}_1\dot{q}_2 l\sin q_2 + l^2\ddot{q}_2)$
  - ⋆ $\frac{\partial T}{\partial q_2} = \frac{\partial}{\partial q_2}\left(m_1\dot{q}_1^2 + \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)\right) = -m_2\dot{q}_1\dot{q}_2 l\sin q_2$
  - ⋆ $\frac{\partial V}{\partial q_2} = \frac{\partial}{\partial q_2}\left(\frac{1}{2}kq_1^2 + mgl\cos q_2\right) = -m_2 gl\sin q_2$
  - ⋆ $F_2 = \left.\frac{\delta W_{nc}}{\delta q_2}\right|_{\delta q_1=0} = \left.\frac{f\delta q_1 - b\dot{q}_1\delta q_1 + \tau\delta q_2}{\delta q_2}\right|_{\delta q_1=0} = \tau$
  - ⋆ T'fore $m_2(\ddot{q}_1 l\cos q_2 - \dot{q}_1\dot{q}_2 l\sin q_2 + l^2\ddot{q}_2) + m_2\dot{q}_1\dot{q}_2 l\sin q_2 + m_2 gl\sin q_2 = \tau$

Therefore $m_2 l\ddot{q}_1\cos q_2 + m_2 l^2\ddot{q}_2 - m_2 lg\sin q_2 = \tau$