Linear State-Space Equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}}_{B} \begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix}$$

$x_1(t), \dots, x_n(t)$ are state-variables

$u_1(t), \dots, u_p(t)$ are system inputs

We can define $\underline{x}(t) := \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ $\quad \underline{u}(t) := \begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix}$

$$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \qquad \text{(State-Equations)}$$

State equations are <u>n linear coupled 1st order</u> diff. eq'ns

The outputs are linear combinations of the state-variables and the inputs.

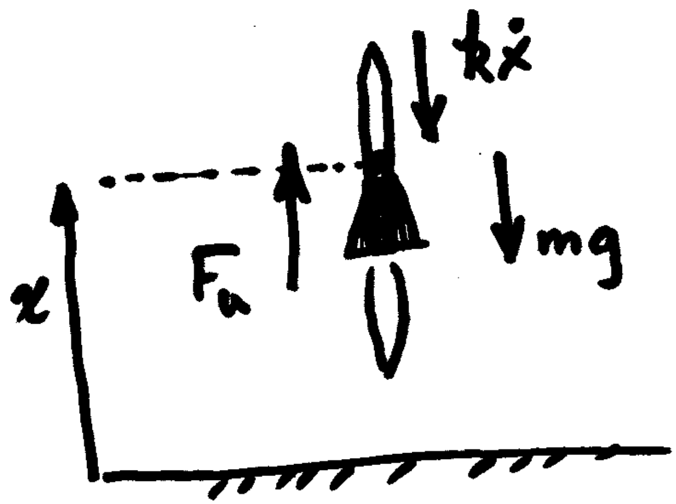
$$\underbrace{\begin{bmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{bmatrix}}_{\underline{y}(t)} = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{q1} & \cdots & c_{qn} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{\underline{x}} + \underbrace{\begin{bmatrix} d_{11} & \cdots & d_{1p} \\ \vdots & & \vdots \\ d_{q1} & \cdots & d_{qp} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix}}_{\underline{u}}$$

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \qquad \text{(Output-Equations)}$$

Together, state equations & output equations are the state-space description of the system

$$\boxed{\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \end{aligned}}$$

<u>Example 1 (Rocket)</u>



F_u Thrust force
mg Weight of rocket
$k\dot{x}$ Air friction force

<u>Newton's law:</u> $F_u - mg - k\dot{x} = m\ddot{x}$

or $m\ddot{x} + k\dot{x} = (F_u - mg)$ 2nd order linear diff. eq'n

dividing by m $\ddot{x} + \frac{k}{m}\dot{x} = \frac{F_u}{m} - g$

Let $x_1 := x$ (Rocket position)
$x_2 := \dot{x}$ (Rocket velocity)

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{x}(t)$$

$$= -\frac{k}{m}\dot{x}(t) + \frac{F_u}{m} - g$$

$$= -\frac{k}{m} x_2(t) + \frac{F_u}{m} - g$$

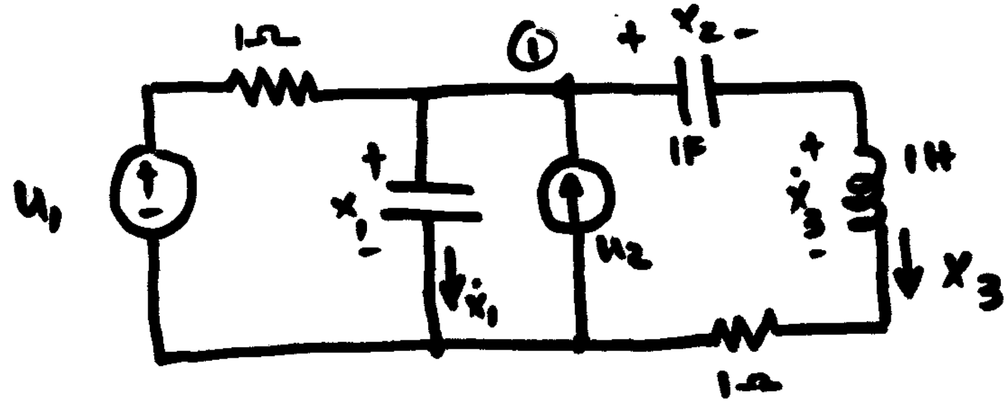
In matrix form:

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}}_{\dot{\underline{x}}(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{k}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\underline{x}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \underbrace{\left(\frac{F_u}{m} - g\right)}_{\underline{u}(t)}$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$ if <u>velocity</u> is output of interest

or $$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$ if position is output of interest

Example 2 (Electric Circuit)



Use: Inductor currents
Capacitor voltages
as state-variables

Summing current at node 1

$$(u_1 - x_1) - \dot{x}_1 + u_2 - x_3 = 0 \quad \cdots \text{(I)}$$

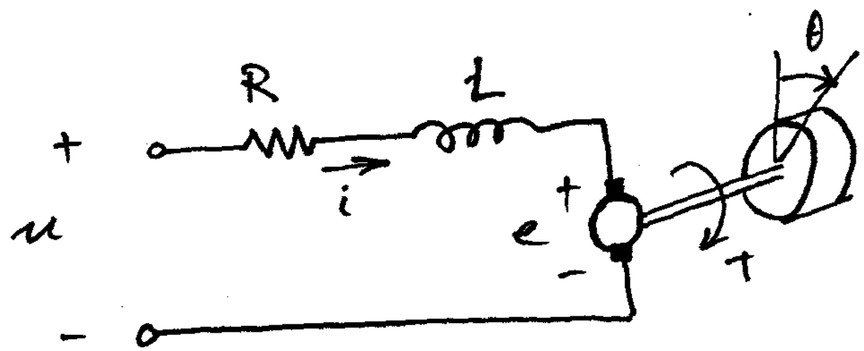
$$\dot{x}_2 = x_3 \quad \cdots \text{(II)}$$

$$\dot{x}_3 + x_3 - x_1 + x_2 = 0 \quad \cdots \text{(III)}$$

In matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Example 3 (Electric Motor)



$$u = Ri + L\frac{di}{dt} + e \implies \frac{di}{dt} = -\frac{R}{L}i - \frac{e}{L} + \frac{u}{L}$$

$$\left.\begin{aligned} T &= I\ddot{\theta} \\ T &= k_1 i \end{aligned}\right\} \implies \ddot{\theta} = \frac{k_1 i}{I}$$

$$e = k_2\dot{\theta}$$

Let $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = i$

Then
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{k_1}{I}x_3$$
$$\dot{x}_3 = -\frac{R}{L}x_3 - \frac{1}{L}k_2x_2 + \frac{u}{L}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & k_1/I\\ 0 & -\frac{k_2}{L} & -\frac{R}{L}\end{bmatrix}}^{A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}}_{B}u$$

If $y = \dot{\theta}$

$$y = \underbrace{[0 \quad 1 \quad 0]}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underset{\sim}{0}\,u$$

Example 4

$$\dddot{y}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = \tilde{u}(t)$$

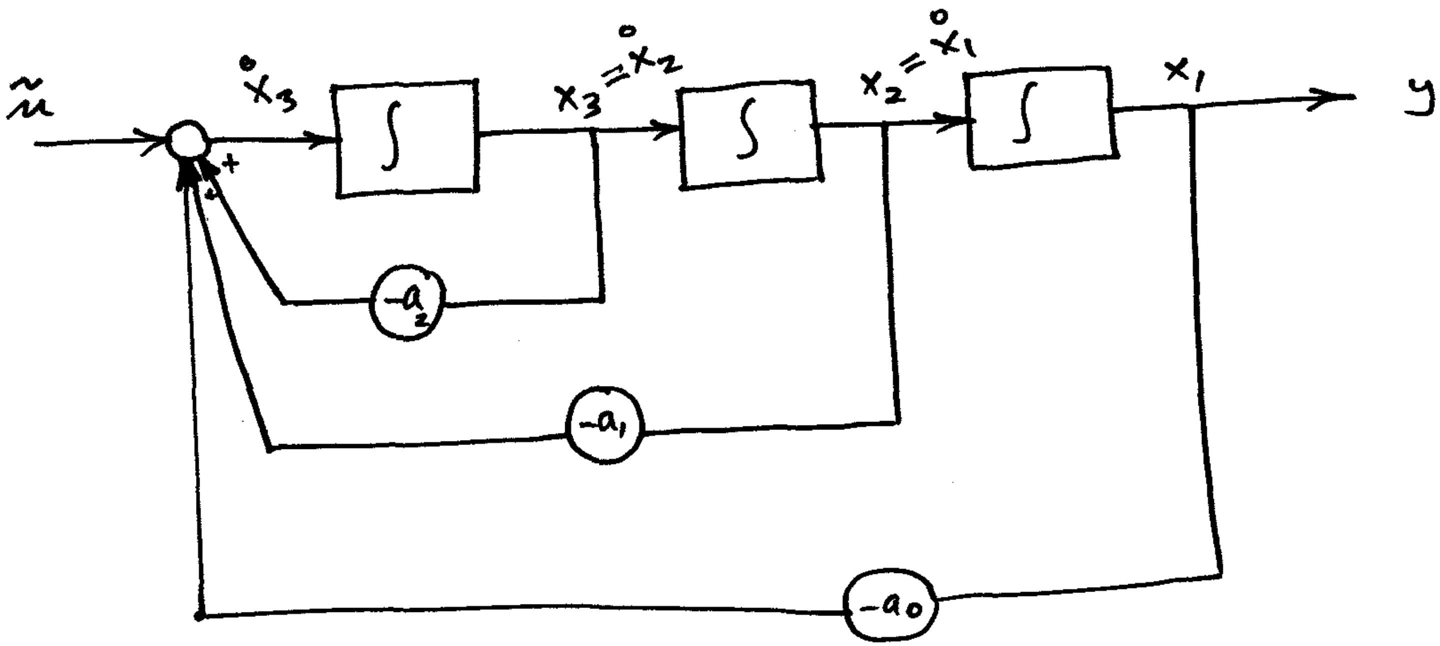
Let
$$x_1 := y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

Then
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = \dddot{y}(t) = -a_2 \ddot{y}(t) - a_1 \dot{y}(t) - a_0 y(t) + \tilde{u}(t)$$
$$= -a_2 x_3 - a_1 x_2 - a_0 x_1 + \tilde{u}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{B} \tilde{u}$$

$$y = \overbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}^{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{0}^{D} \tilde{u}$$

Analog Computer Simulation



Example 5 :

$$y(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \, u(s)$$

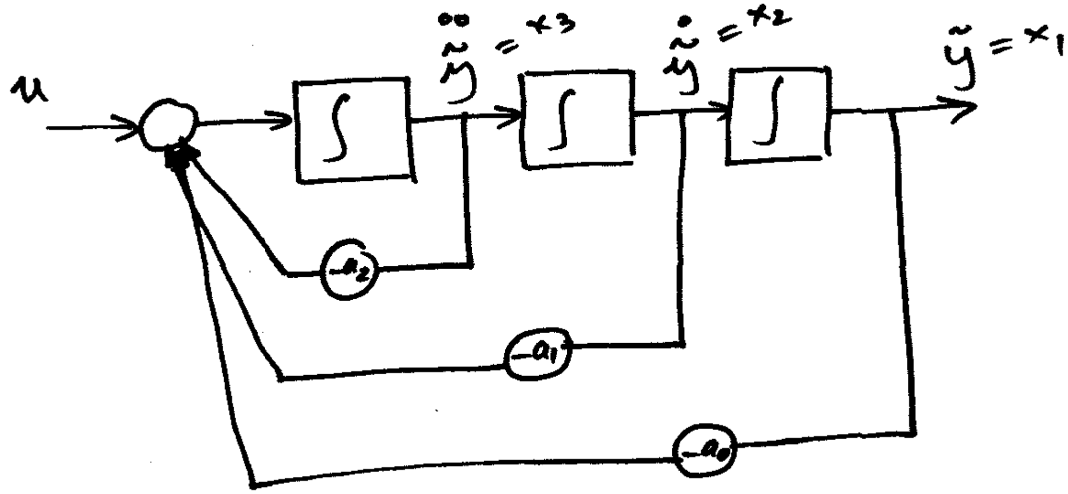
Or equivalently, $\dddot{y}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t)$

To get a state-space representation, we solve the problem in 2 steps:

(1) Letting $\tilde{y}(s) := \dfrac{u(s)}{s^3 + a_2 s^2 + a_1 s + a_0}$, we can obtain a state-space description with $\tilde{y}(s)$ as the output:

$$\left(s^3 + a_2 s^2 + a_1 s + a_0\right) \tilde{y}(s) = u(s) \qquad \text{or}$$

$$\dddot{\tilde{y}}(t) + a_2 \ddot{\tilde{y}}(t) + a_1 \dot{\tilde{y}}(t) + a_0 \tilde{y}(t) = u(t)$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

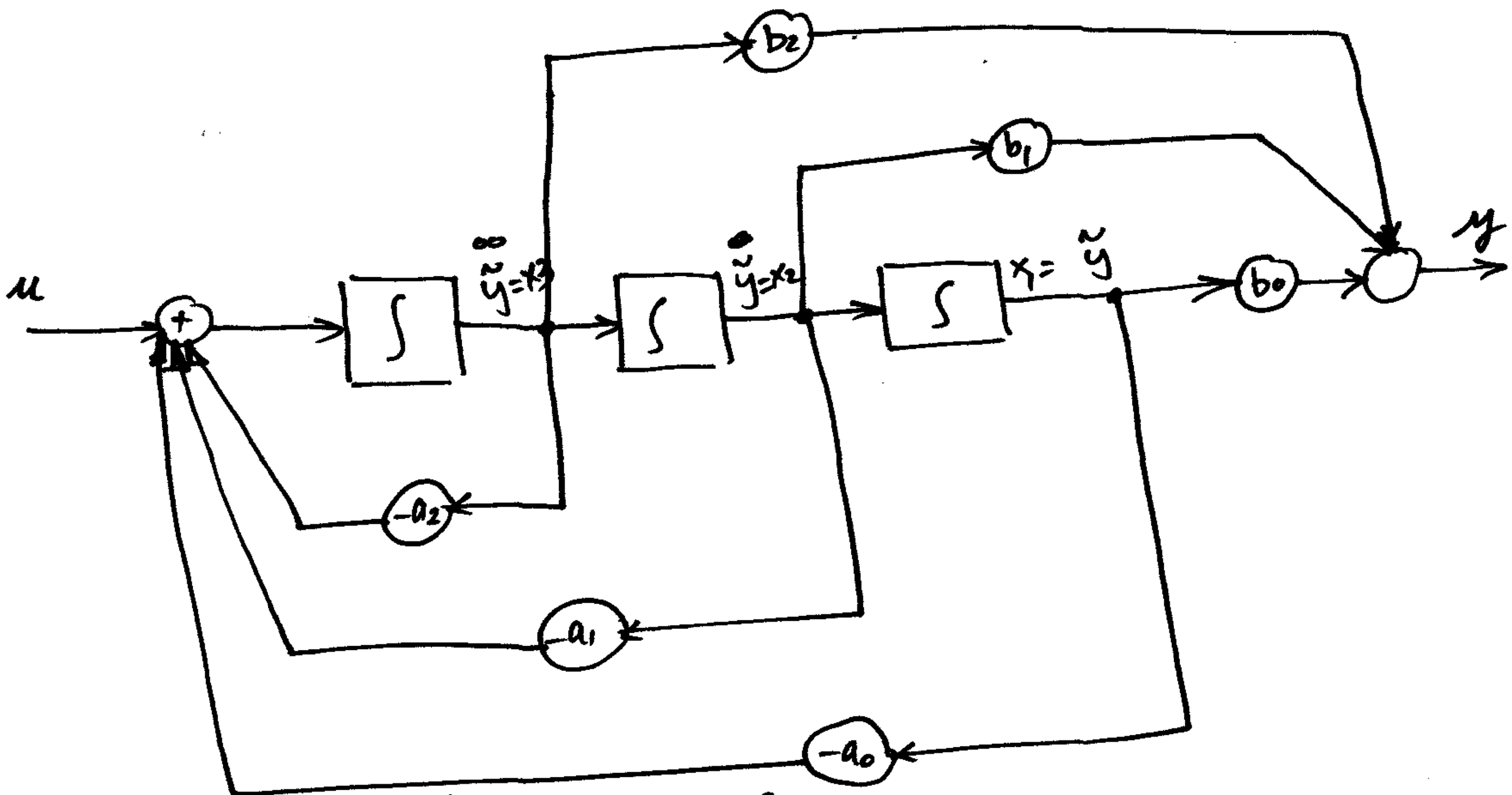
(2) $y(s) = (b_2 s^2 + b_1 s + b_0)\tilde{y}(s)$ or

$$y(t) = b_2 \ddot{\tilde{y}}(t) + b_1 \dot{\tilde{y}}(t) + b_0 \tilde{y}(t)$$

$$= b_2 x_3 + b_1 x_2 + b_0 x_1$$

$$y(t) = \underbrace{\begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(11)



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overset{B}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} u$$

$$y = \underbrace{\begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{[0]}_{D} u$$

Linear Systems as approximations to nonlinear Systems

Given:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, \dots, x_n, u_1, \dots, u_p) \\ \vdots \\ f_n(x_1, \dots, x_n, u_1, \dots, u_p) \end{bmatrix} \quad \text{(State eq's)}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \begin{bmatrix} g_1(x_1, \dots, x_n, u_1, \dots, u_p) \\ \vdots \\ g_q = (x_1, \dots, x_n, u_1, \dots, u_p) \end{bmatrix} \quad \text{(Output eq's)}$$

Or

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \quad \text{(State eq'n)}$$

$$\underline{y} = \underline{g}(\underline{x}, \underline{u}) \quad \text{(Output eq'n)}$$

$(\underline{x}_0^{\in\mathbb{R}}, \underline{u}_0^{\in\mathbb{R}})$ is an equilibrium solution if

$$\underline{f}(\underline{x}_0, \underline{u}_0) = 0$$

In this case $\underline{x}(t) \equiv \underline{x}_0$ and $\underline{u}(t) \equiv u_0$ solve the differential equation (Constant Solution)

We can look at solutions close the equilibrium

Let $x(t) := x_0 + \delta x(t)$

$u(t) := u_0 + \delta u(t)$

$$\mathring{x}_i(t) = \dot{\delta x}_i = f_i(\underline{x}, \underline{u})$$

$$= \underbrace{\cancel{f_i(\underline{x}_0, \underline{u}_0)}}_{0} + \left.\frac{\partial f_i}{\partial x_1}\right|_{\substack{x_0\\u_0}} \delta x_1 + \cdots + \left.\frac{\partial f_i}{\partial x_n}\right|_{\substack{x_0\\u_0}} \delta x_n$$

$$+ \left.\frac{\partial f_i}{\partial u_1}\right|_{\substack{x_0\\u_0}} \delta u_1 + \cdots + \left.\frac{\partial f_i}{\partial u_p}\right|_{\substack{x_0\\u_0}} \delta u_p$$

$+$ higher order terms

Similarly for each output y_i ....

In matrix form:

$$\begin{bmatrix} \delta \dot{x}_1 \\ \vdots \\ \delta \dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{\substack{x_0 \\ u_0}}}_{A} \begin{bmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}\Bigg|_{\substack{x_0 \\ u_0}}}_{B} \begin{bmatrix} \delta u_1 \\ \vdots \\ \delta u_p \end{bmatrix}$$

+ higher order terms

Ignoring H.O.T. we have

$$\boxed{\begin{aligned} &\delta \dot{\underline{x}} \cong A\, \underline{\delta x} + B\, \underline{\delta u} \qquad &\text{(Linearized Input eq's)} \\ &\underline{\delta y} \cong C\, \underline{\delta x} + D\, \underline{\delta u} \qquad &\text{(Linearized Output eq's)} \end{aligned}}$$