Bode-plot and stability

Sunday, November 15, 2009
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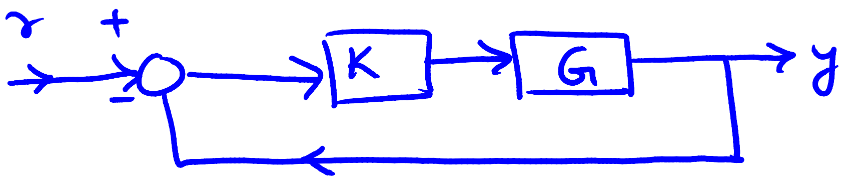
Consider the unity negative feedback interconnection given below



We will first analyze the case when K is a real and positive constant. and thus $K(s) = K$ and thus $\frac{n_K}{d_K}$ with $n_K = K$ and $d_K = 1$ forms a coprime representation of the controller K.

(*) The characteristic polynomial is given by

$$n_G n_K + d_G d_K$$

$$= n_G K + d_G$$ where $\frac{n_G}{d_G}$ is a coprime representation of G.

K goes to zero

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Thus, the stability of the feedback interconnection is governed by the the roots of the polynomial given by

$$Kn_G + d_G = 0$$

Clearly as $K \to 0$ the polynomial

$$Kn_G + d_G \approx d_G(s)$$

and thus,

→ as $K \to 0$, the roots of the characteristic polynomial are given by the poles of the transfer function G (i.e. roots of d_G).

⊕ Now lets consider what happens when $K \to \infty$.

K goes to infinity

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The characteristic polynomial

$K n_G + d_G$ can be rewritten

as $n_G + \frac{1}{K} d_G$

and as $K \to \infty$

$$n_G + \frac{d_G}{K} \simeq n_G$$ and

thus, the roots of the characteristic polynomial go toward the open-loop zeros of the plant G (i.e. roots of n_G).

Also, in most cases

$$\deg(n_G) = m$$
$$\deg(d_G) = n$$

with $n - m = q > 0$

poles at infinity

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Thus, the characteristic polynomial roots are obtained by

$$d_G + K n_G = 0$$

$$\Leftrightarrow \quad 1 + K \frac{n_G}{d_G} = 0$$

$$\Leftrightarrow \quad \frac{n_G}{d_G} = -\frac{1}{K} \quad \text{-----} \; (*)$$

clearly as $K \to \infty$; $-1/K \to 0$

Thus, for $\bar{s}$ to be solution of (*)

$$\frac{n_G(\bar{s})}{d_G(\bar{s})} = 0$$

that happens at every root of $n_G(s)$.

However this condition can also hold as $s \to \infty$. Note that as $s \to \infty$

poles at infinity

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and assuming

$$n_u(s) = \alpha\left(s^m + a_{m-1}s^{m-1} + \dots + a_0 s^0\right)$$

$$d_u(s) = s^n + b_{n-1}s^{n-1} + \dots + b_0 s^0$$

we have

$$\frac{n_u(s)}{d_u(s)} = \alpha\left[\frac{s^m + a_{m-1}s^{m-1} + \dots + a_0}{s^n + b_{n-1}s^{n-1} + \dots + b_0}\right]$$

$$= \alpha \frac{s^m}{s^n} \frac{\left[1 + \frac{a_{m-1}s^{m-1}}{s^m} + \dots + \frac{a}{s^m}\right]}{\left[1 + b_{n-1}\frac{s^{n-1}}{s^n} + \dots + \frac{b_0}{s^n}\right]}$$

$$\cong \frac{\alpha}{s^q} \qquad \text{when } |s| \gg 1.$$

and thus, $\frac{\alpha}{s^q} \to 0$ as $s \to \infty$ also

and thus there are q roots of the characteristic polynomial at $|s| = \infty$ in addition to m roots at the zeros of G for a total of n roots of the characteristic polynomial.

poles at infinity

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In summary as $K \to \infty$
the roots of the characteristic
polynomial move toward the
m zeros of G (roots of the polynomial
$N(s)$) and $(n-m) = q$ values at $|s| = \infty$.

Typical Scenario

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Lets assume that G is stable transfer function.

Thus, d_G has roots only in the strict L.H.P $[\{s \mid Re(s) < 0\}]$.

(*) Clearly when $k = 0$, the characteristic polynomial has all roots stable [as the characteristic polynomial $n_G k + d_G = d_G$] and the closed-loop system is stable.

(*) Also, typically the transfer function G is such that, at least some roots of the characteristic polynomial move into the RHP. as k is increased

Roots on the imaginary axis

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Thus, the typical scenario is

- For K small the interconnection is stable and all closed-loop poles (the roots of the characteristic polynomial) are in the LHP

- For K very large at, least some closed-loop poles are located in the RHP

- From continuity arguments it follows that there is a value of K for which

$$n_G(j\omega)\, K + d_G(j\omega) = 0 \text{ for some}$$

values of ω. Thus, the closed-loop poles cross the $j\omega$ axis as K is increased

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from low values of K to large values of K, with an intermediate value of K; say K_{cr} at which the characteristic polynomial has root on the $j\omega$ axis.

Thus,

$K < K_{cr}$, the interconnection is stable

and

$K > K_{cr}$ the interconnection is unstable

with the characteristic polynomial having imaginary axis zeros for.
$K = K_{cr}$.

The above condition is utilized by Bode plots to determine if $K < K_{cr}$ or $K > K_{cr}$

Critical Gain value

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Note that for imaginary axis zero of the characteristic polynomial there should be a frequency ω such that

$$K_{cr} n_G(j\omega) + d_G(j\omega) = 0$$

i.e.

$$\frac{K_{cr} n_G(j\omega)}{d_G(j\omega)} = -1$$

Thus,

$$K_{cr} G(j\omega) = -1$$

Thus,

$$|K_{cr}| |G(j\omega)| = 1$$

and

$$\angle K_{cr} G(j\omega) = -180$$

$$\Rightarrow \quad \angle G(j\omega) = -180.$$

Thus, the Bode plot at K_{cr} of $K_{cr} G$ will look like

Critical Gain Value

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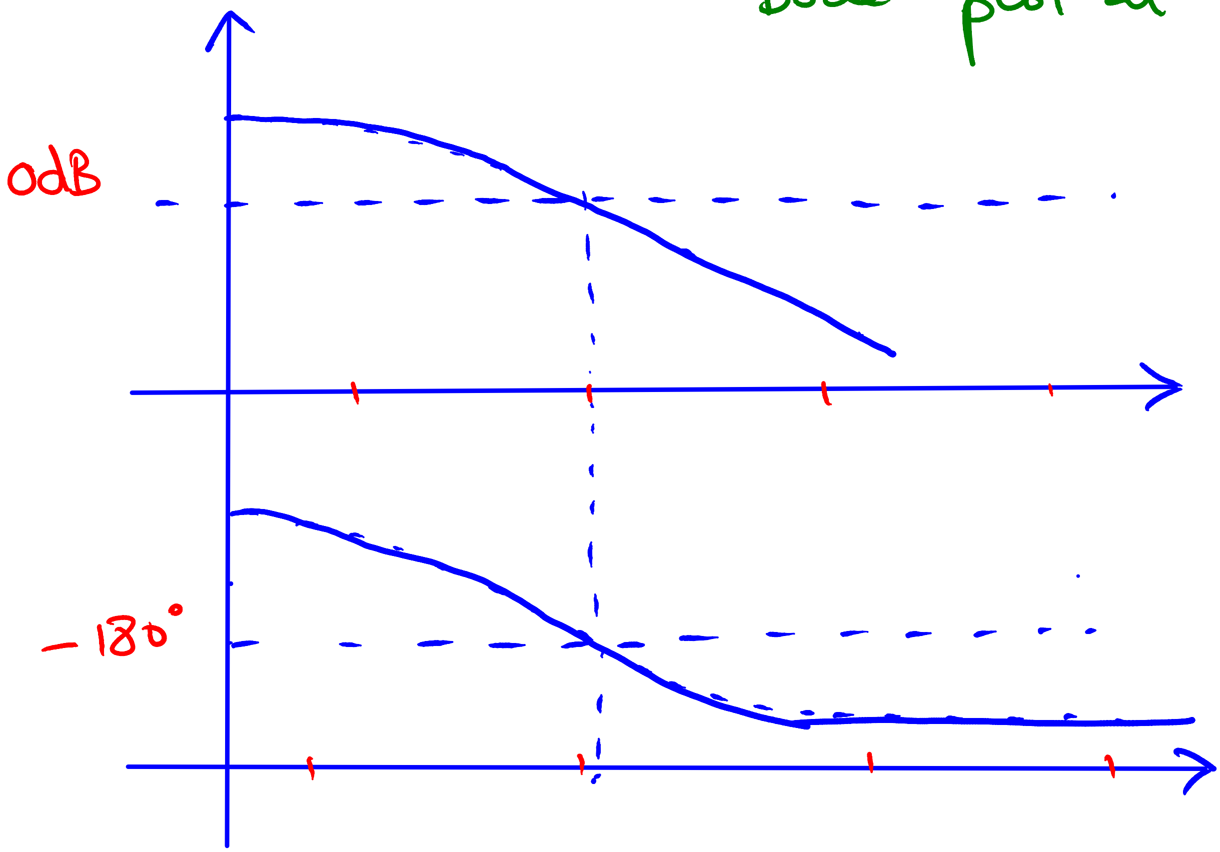
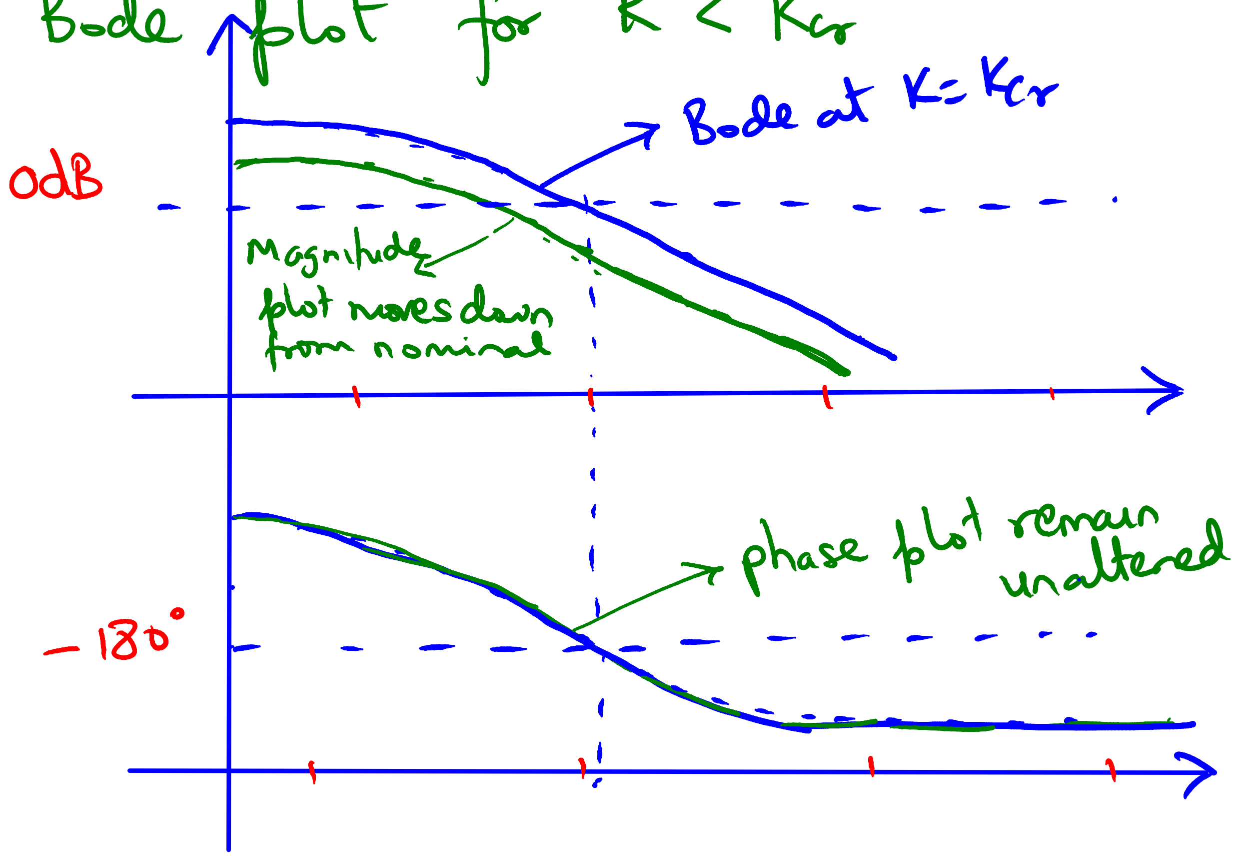
Bode plot at K_{cr}.



Bode plot for $K < K_{cr}$



Value above critical value

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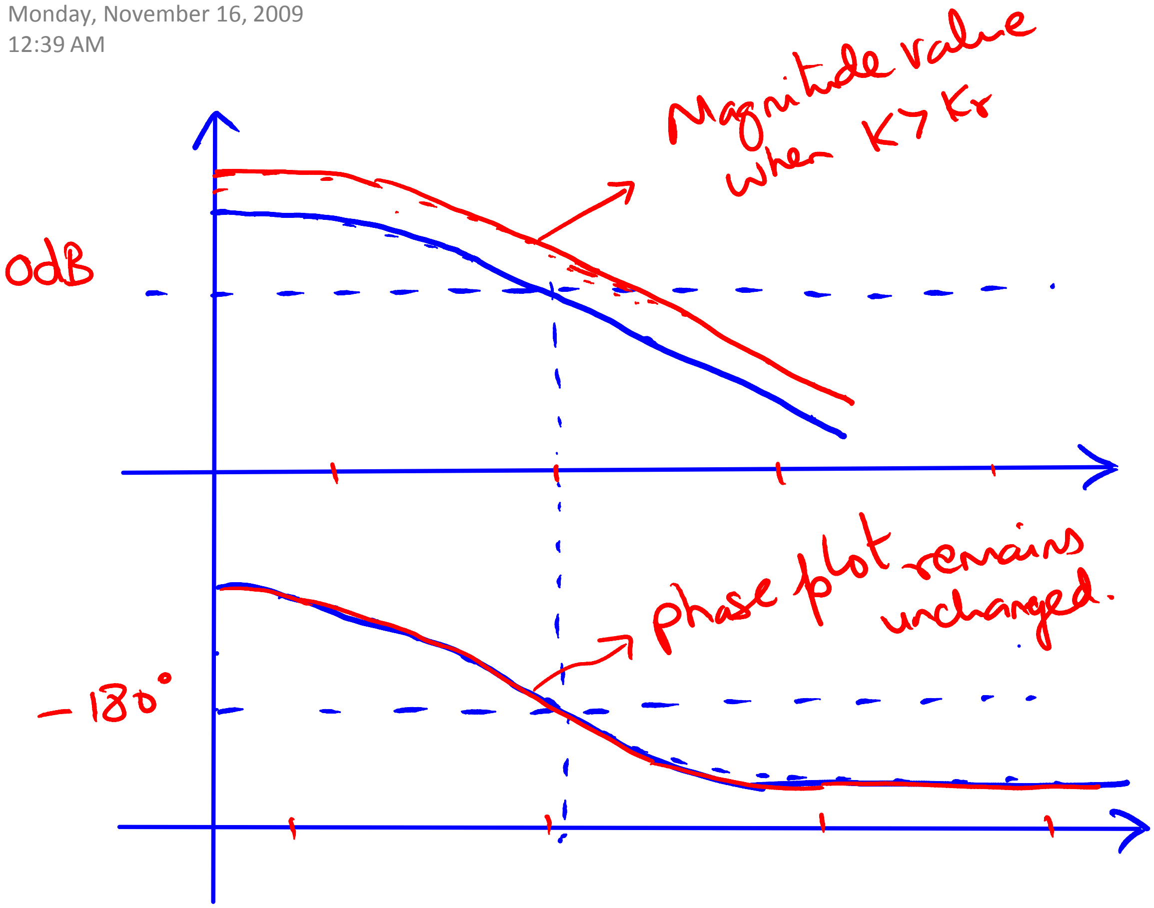

Summary of a typical Scenario

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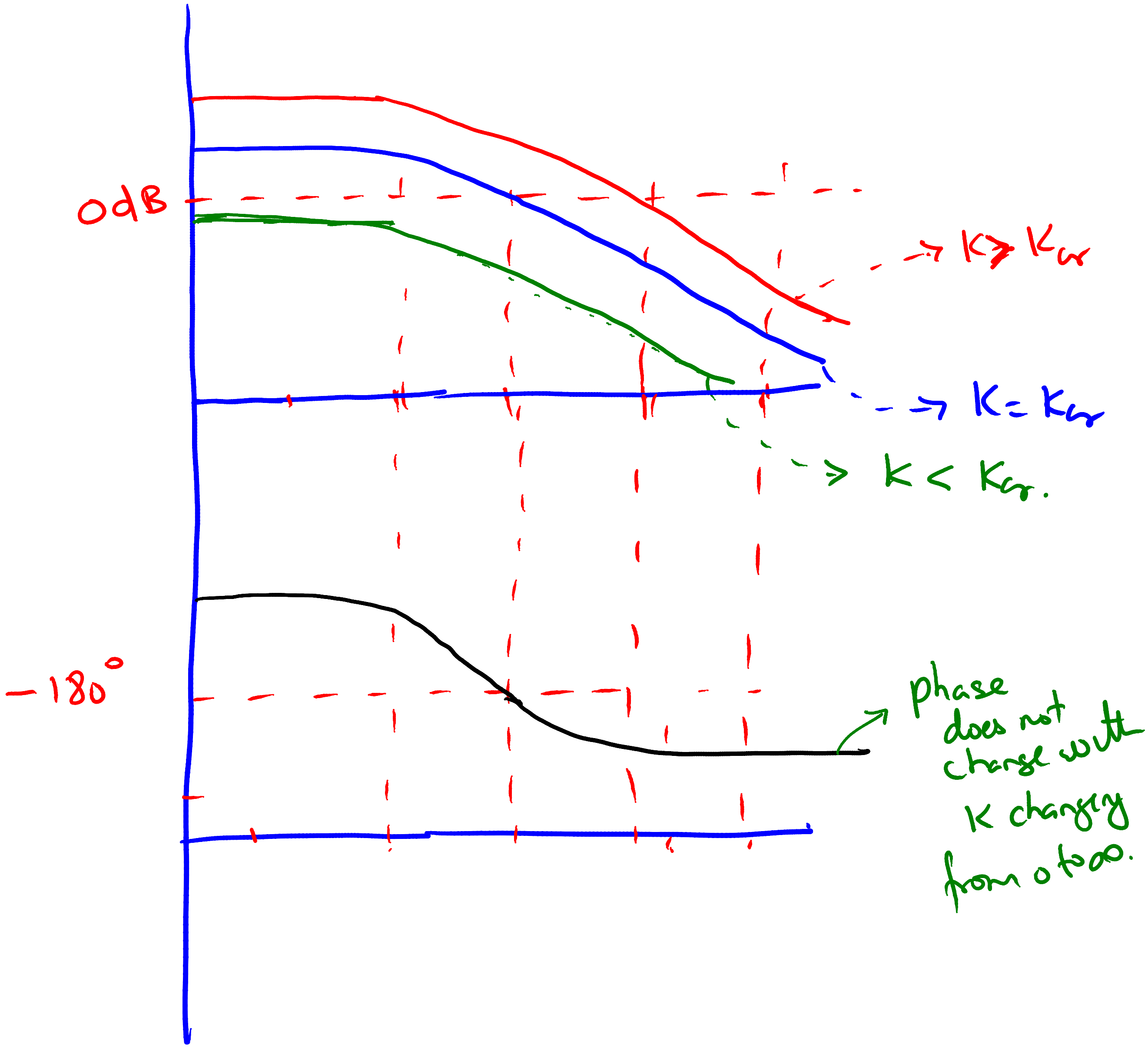

→ Note that when $K > K_{cr}$ then the typical interconnection is unstable

→ Note that when $K < K_{cr}$ then the typical interconnection is stable.

GAIN MARGIN

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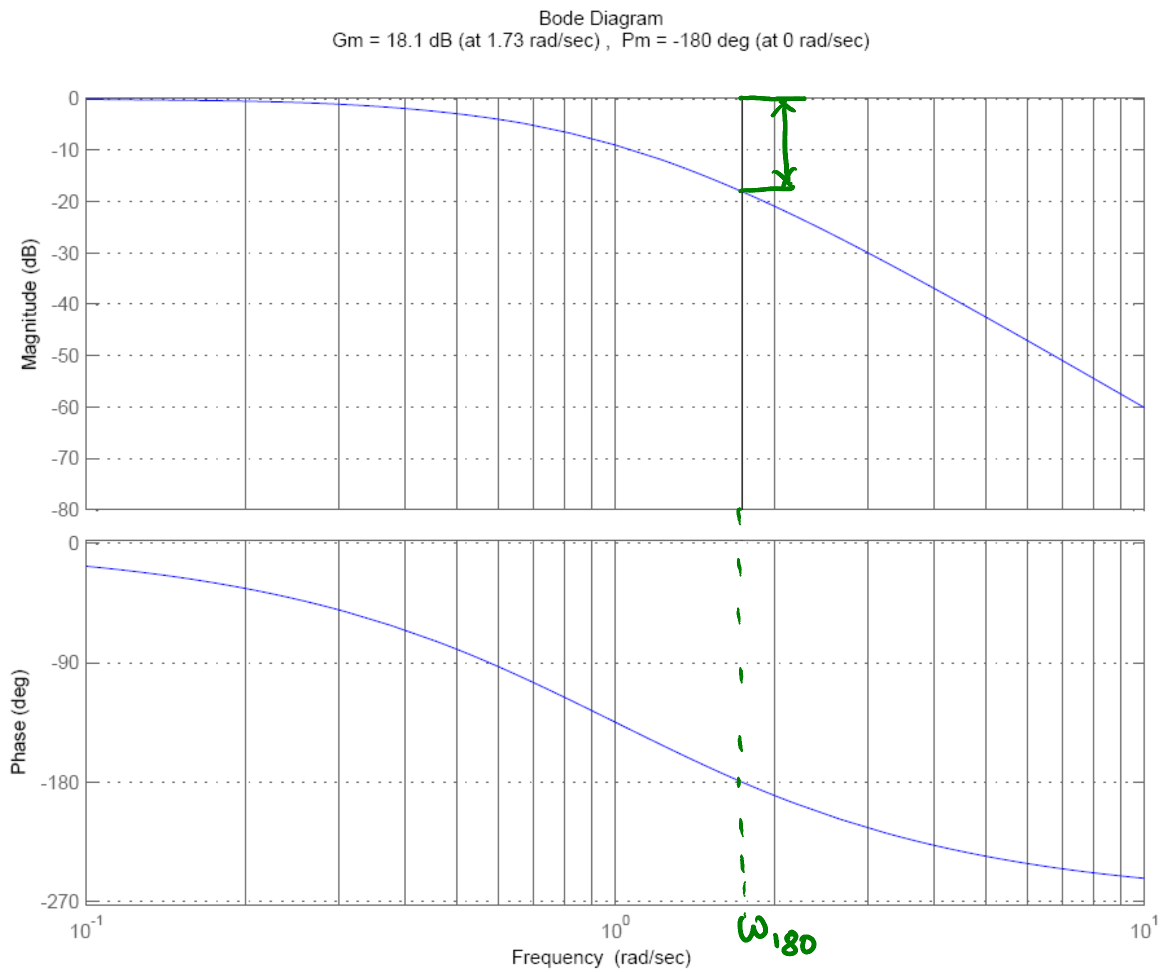
How much can the gain be increased without rendering the unity negative feedback interconnection unstable is characterized by the Gain Margin



Let ω_{180} be the frequency such that $\angle L(j\omega_{180}) = -180°$. Then

$$\boxed{\text{Gain Margin} = GM = -20\lg_{10}|L(j\omega_{180})|}$$

Gain Margin

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- If the Gain Margin is positive a typical unity negative feedback interconnection is stable.

- If the GM is negative then a typical unity negative feedback interconnection is unstable.

Phase Margin

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Note that the closed-loop system (the unity negative feedback interconnection) has a closed loop pole on the imaginary $j\omega$ axis if

$$1 + KG(j\omega) = 0 \quad \text{for some } \omega$$

i.e. when

$$1 + L(j\omega) = 0 \quad \text{for some } \omega$$

$$\Leftrightarrow \quad \angle L(j\omega) = -180 \quad \text{for some } \omega.$$

$$|L(j\omega)| = 1$$

Let ω_{gc} be the value of ω such that (ω_{gc} is called the gain crossover frequency)

$$|L(j\omega_{gc})| = 1$$

and suppose the phase of L at ω_{gc} is $\angle L(j\omega_{gc})$. Then

$$\boxed{PM = \angle L(j\omega_{gc}) - (-180) = 180 + \angle L(j\omega_{gc})}$$

Phase Margin

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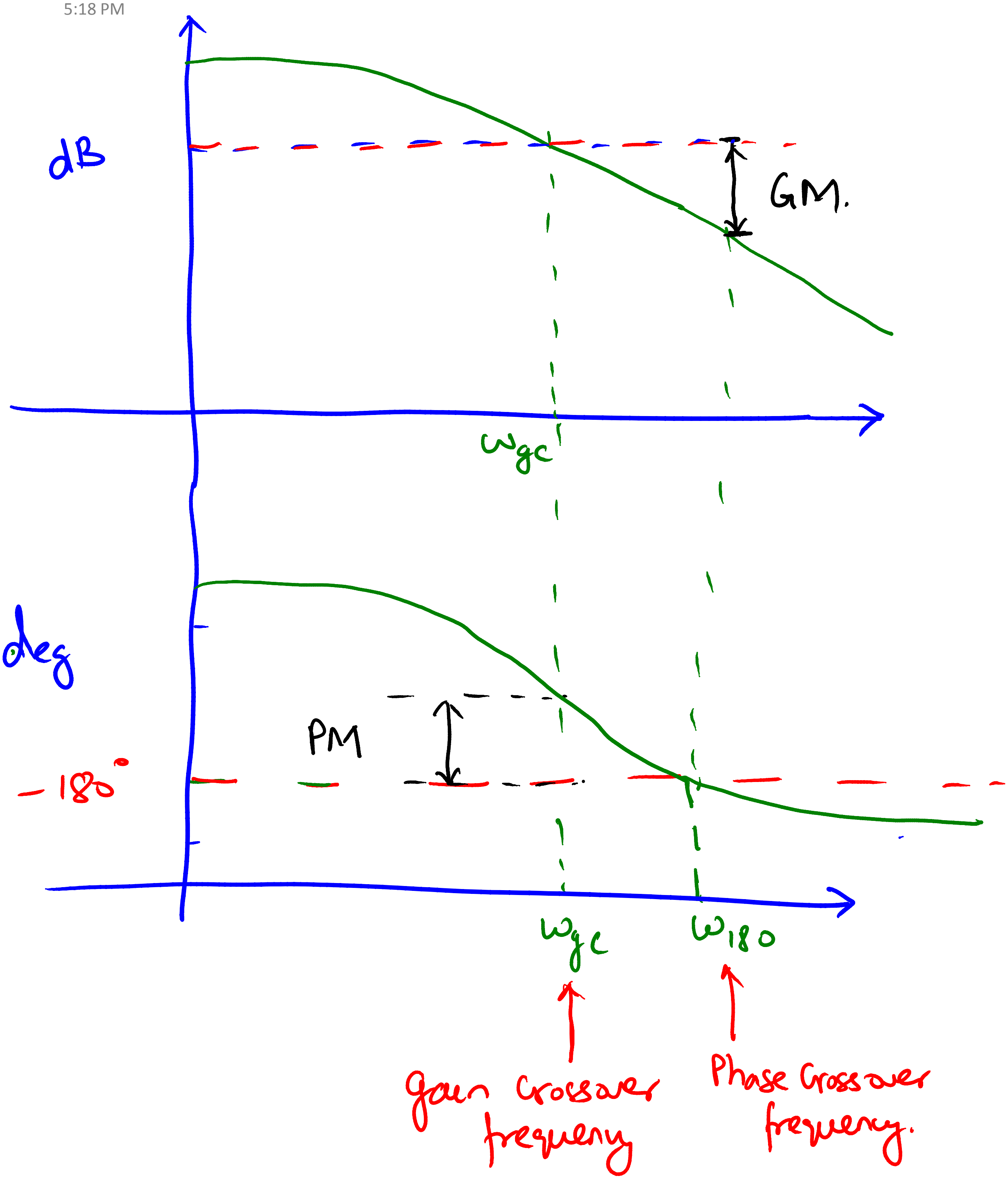

Phase Margin

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Thus, again (for a typical system)

- If the phase Margin is positive the closed-loop system is stable
- If the phase Margin is negative the closed-loop system is unstable.

Frequency Domain Specifications

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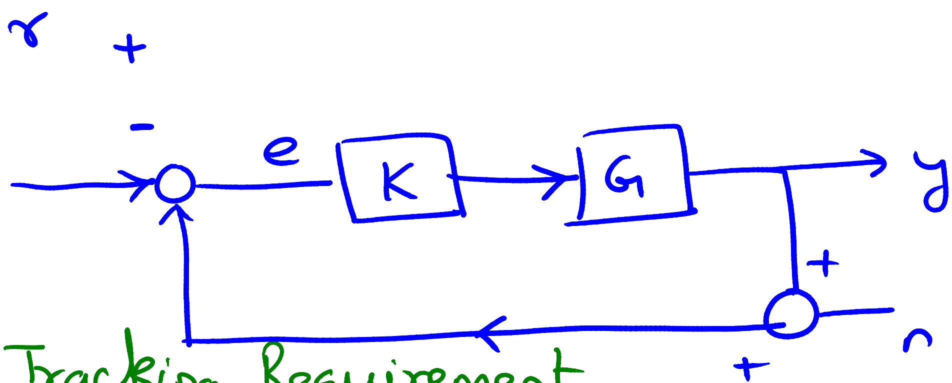
Consider the unity negative feedback closed-loop system



Tracking Requirement

→ In many controls applications it is desired that the output y "tracks" r i.e. the error between y and r is required to be small.

→ Also, it is typical that the tracking is required not only for a specific input but

Frequency domain specifications

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over possibly many kinds of input with the characterization that the class of input signals to be tracked have bandwidth content withing a certain frequency range ω_B.

Thus, if $r(t)$ is the reference trajectory to be tracked then its Fourier transform

$$r(j\omega) = \int r(t) e^{-j\omega t} dt$$

is such that

$$\left|\frac{r(j\omega)}{r(0)}\right| << 1 \quad \forall \omega > \omega_B.$$

[note that we are using $r(t)$ and $r(j\omega)$ to represent the time domain and the frequency domain quantities respectively]

Tracking Requirement.

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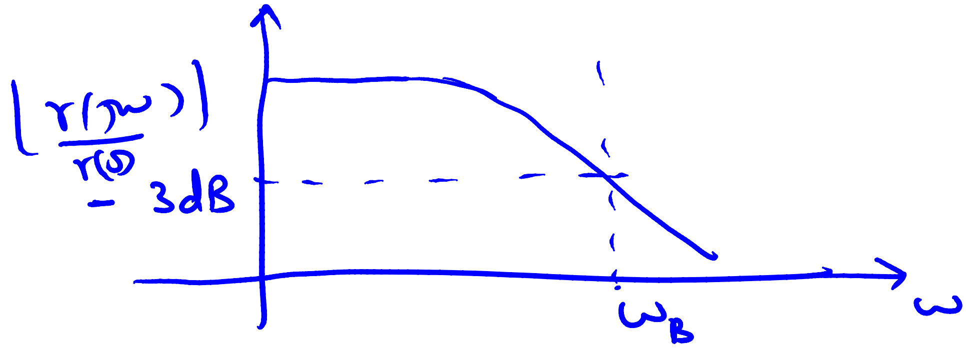
Thus, it is known that $r(j\omega)$ is such that



Thus, for $y(t)$ to track $r(t)$; we need the error $e(t) = r(t) - y(t)$ to be small. As $r(t)$ has a Fourier transform $r(j\omega)$ with $\left|\frac{r(j\omega)}{r(0)}\right|$ negligible beyond ω_B we need for good tracking

$e(j\omega)$ to be small for all $\omega \leq \omega_B$

Now $$e(s) = \frac{1}{1+GK} r(s)$$

Tracking Requirement

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and thus we need

$$\frac{e(j\omega)}{r(j\omega)} = \frac{1}{1 + G(j\omega)K(j\omega)}$$

to be small for all $\omega \le \omega_B$.

Sensitivity Transfer function:

The transfer function

$$S = (1 + GK)^{-1}$$ is called

the sensitivity transfer function.

Thus, the tracking requirement translates to having

$$|S(j\omega)| \text{ to be small}$$

for all $\omega \le \omega_B$.

ω_B is called the tracking bandwidth.

Noise Rejection

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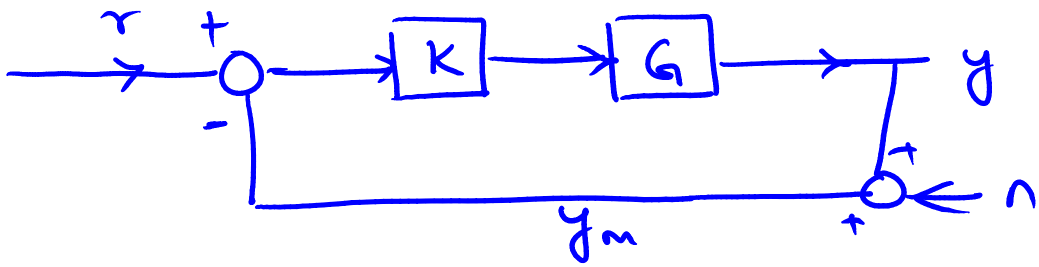

Any measurement procedure is not precise; every sensor introduces noise, n. However, it is not desired that the noise n affect the output y.

The transfer function between noise n to the output y is given

$$\frac{-GK}{1+GK}.$$

and thus, it is typically desired that this transfer function is small beyond a

Noise Rejection

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frequency ω_{BT}. 1.

<u>Complimentary Sensitivity transfer function.</u>

$$T = \frac{GK}{1+GK}$$

is called the complimentary transfer function.

Thus, we need

$|T(j\omega)|$ to be small

beyond $\omega > \omega_{BT}$.

Summary

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In summary for

(a) Tracking we need

$$S = \frac{1}{1+GK} = \frac{1}{1+L}$$ to

be small in the frequency

range $\omega \le \omega_B$

(b) For noise rejection we need

$$T = \frac{GK}{1+GK} = \frac{L}{1+L}$$ to

be small in the frequency

range $\omega \geq \omega_{BT}$.

A fundamental limitation

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Note that

$$S(j\omega) = \frac{1}{1+L(j\omega)}$$

and $$T(j\omega) = \frac{L(j\omega)}{1+L(j\omega)}$$

and therefore

$$S(j\omega) + T(j\omega) = \frac{1}{1+L(j\omega)} + \frac{L(j\omega)}{1+L(j\omega)}$$

$$= \frac{1+L(j\omega)}{1+L(j\omega)}$$

$$= 1.$$

and thus, at the same frequency ω it is not possible to have both $S(j\omega)$ and $T(j\omega)$ small. Fortunately the requirements on S and T are possibly on different frequency ranges $[0, \omega_B)$ and $[\omega_{BT}, \infty)$ respectively.

Translating closed-loop objectives to open loop tf

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Here we will translate the conditions on the closed loop map to the open-loop transfer function L.

Note that

we want

$$|S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| \text{ to be small}$$

in the frequency region $\omega \in [0, \omega_B]$

This implies we need

$$|L(j\omega)| \text{ to be large}$$

in the frequency region $[0, \omega_B]$

Typically, we need

$$20\log_{10}|L(j\omega)| \geq +3\text{dB} \quad \text{for } \omega \in [0, \omega_B]$$

Closed-loop objective to open-loop tf.

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The noise-rejection objective is that the complimentary transfer function

$$|T(j\omega)| = \left|\frac{L(j\omega)}{1+L(j\omega)}\right|$$

to be small in $\omega \in [\omega_{BT}, \infty)$.

i.e

$$|T(j\omega)| = \left|\frac{1}{\frac{1}{L(j\omega)}+1}\right| \text{ to}$$

be small in $\omega \in [\omega_{BT}, \infty)$

Thus, we require $|L(j\omega)|$ to be small in the range $\omega \in [\omega_{BT}, \infty)$.

Typically it is needed that

$20\log_{10}|L(j\omega)| \leq -3dB$ for $\omega \geq \omega_{BT}$.

Specifications on the open-loop tf.

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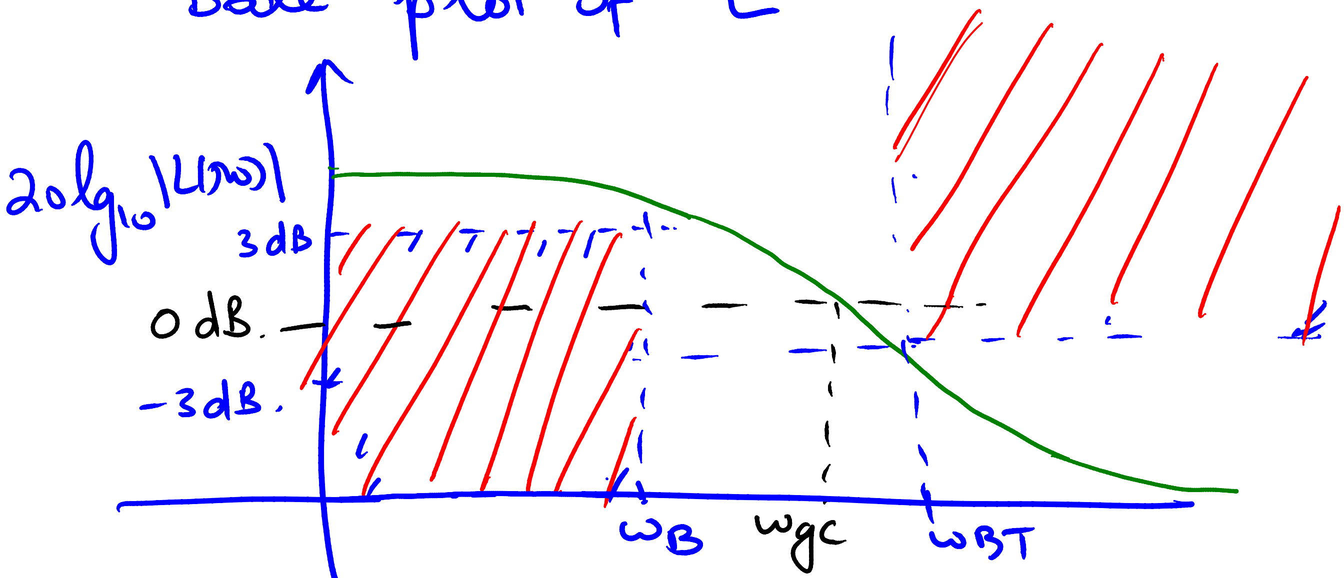
In Summary we need the following magnitude part of the Bode plot of L



→ The red region has to be avoided by the transfer function $L(j\omega)$.

Summary

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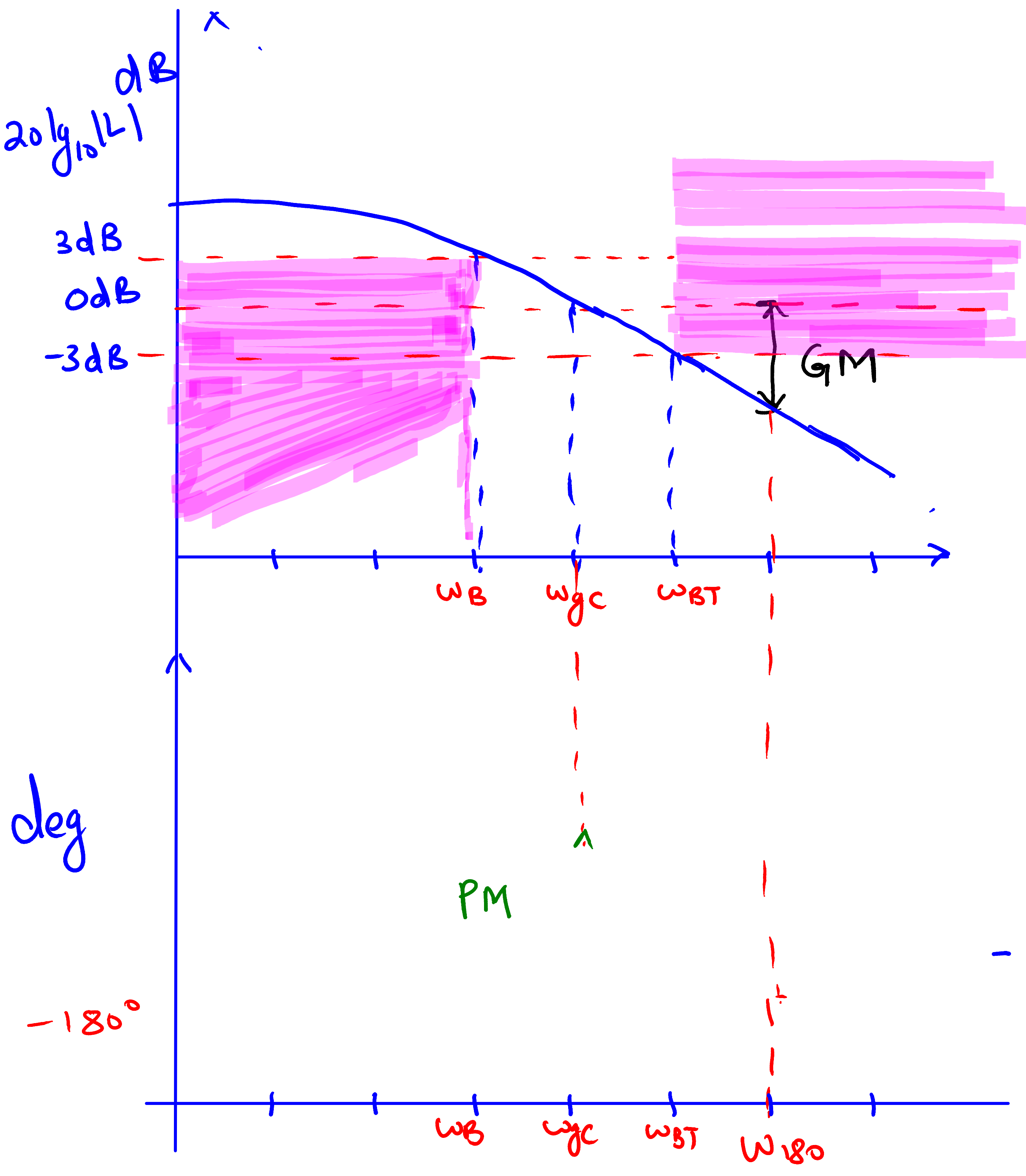

Time-domain Specifications

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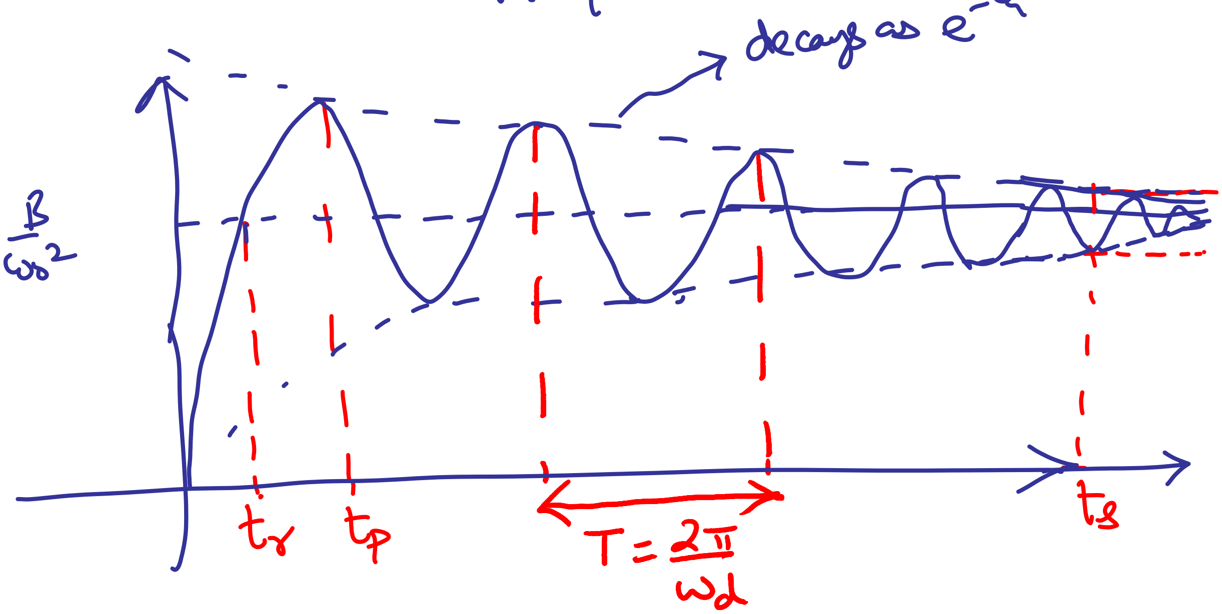
Plots of Step response of

$$\frac{B}{s^2+2\xi\omega_0 s+\omega_0^2}$$

given by

$$y(t)=\frac{B}{\omega_0^2}\left[1-\frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\phi)\right]$$



t_r = rise time: the first time the output reaches the steady state value $y_{ss}=B/\omega_0^2$

t_p = peak time; the time the maximum is reached

t_s = settling time: the time after which output remains within 2% of the steady state y_{ss}

y_{ss} = steady state value B/ω_0^2

$$M_p=\frac{y(t_p)-y_{ss}}{y_{ss}}$$

Time domain Specifications

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→ $y_{ss} = B/\omega_0^2$

→ $t_p = \dfrac{\pi}{\omega_d}$; $\omega_d = \omega_0\sqrt{1-\xi^2}$

→ $t_r = \dfrac{\pi - \phi}{\omega_d}$; $\phi = \cos^{-1}\xi$.

→ $t_s = \dfrac{4}{\xi\omega_0}$; 2% settling time

→ $M_p = e^{\left(-\pi\xi/\sqrt{1-\xi^2}\right)}$

Second order prototype

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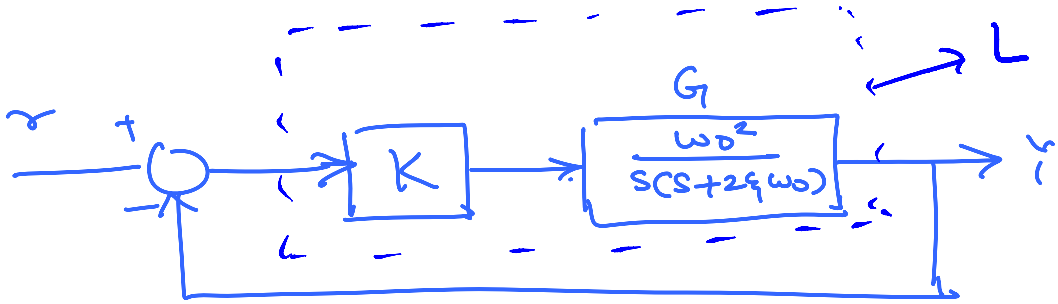

→ open-loop transfer function

$$L = \frac{K\omega_0^2}{s^2 + 2\xi\omega_0 s}$$

→ closed-loop transfer function from $r \mapsto y$ is

$$\frac{L}{1+L} = \frac{K\omega_0^2}{s^2 + 2\xi\omega_0 s + K\omega_0^2}$$

→ For $K=1$ we have the closed-loop $r \mapsto y$ have the prototype second-order transfer function

$$\frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2}.$$

Magnitude of the open-loop L

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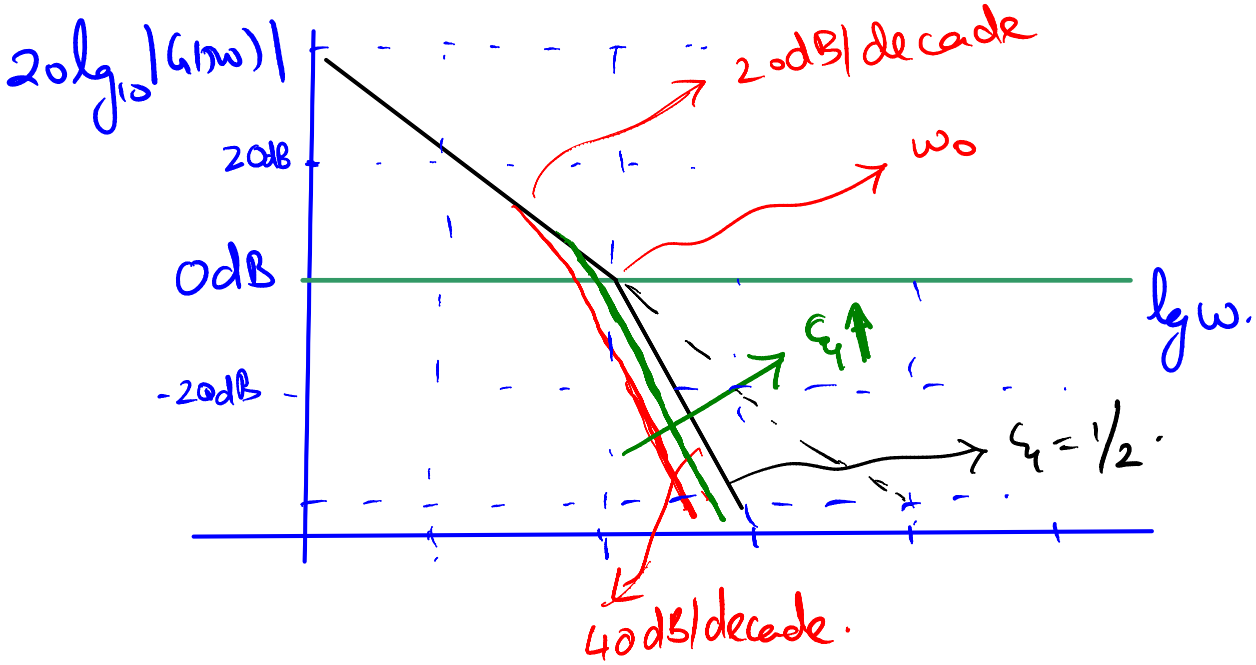

→ A good thumb rule is that the gain cross-over frequency is

$$\omega_{gc} = \frac{\omega_0}{1.4}$$

– A good thumb rule is that the closed-loop bandwidth is $(1.2 - 1.6)\,\omega_{gc}$.

Phase margin of second order prototype

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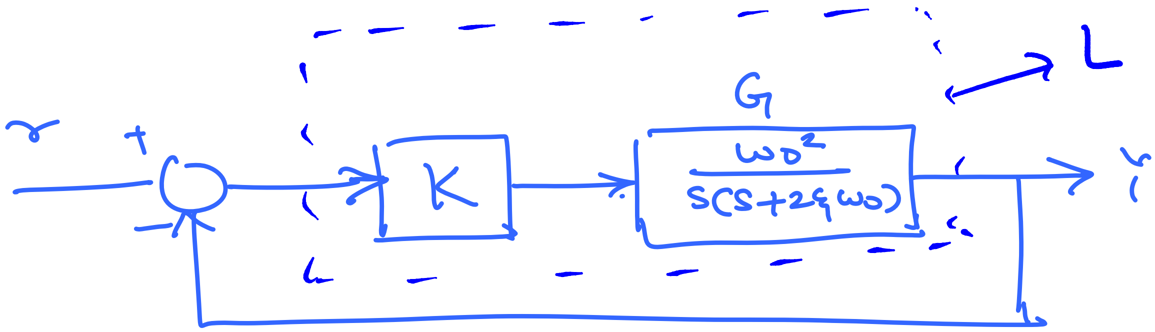
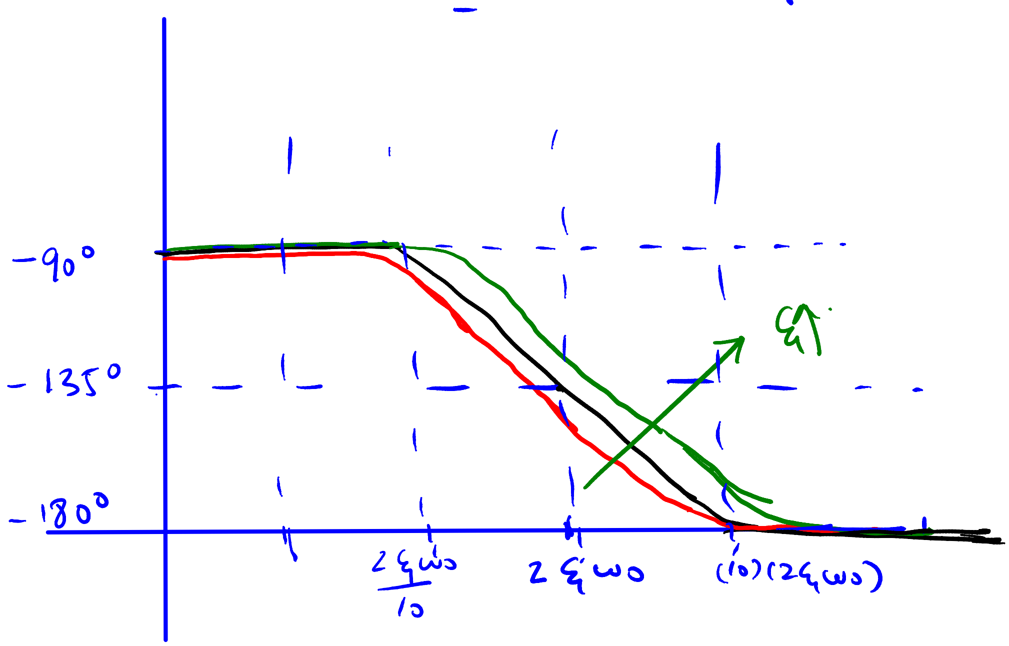


→ The open-loop transfer function

$$L = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s}$$

$$\angle G(j\omega) = \angle \frac{\omega_0}{j\omega(j\omega + 2\zeta\omega_0)}$$

$$= -\pi/2 - \operatorname{atan}\left(\frac{\omega}{2\zeta\omega_0}\right)$$



Phase Margin of a Second order prototype.

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$$\text{Phase margin} = \pi + \angle G(j\omega_{gc})$$

$$= \pi - \frac{\pi}{2} - \operatorname{atan}\left(\frac{\omega_{gc}}{2\zeta\omega_0}\right)$$

$$= \frac{\pi}{2} - \operatorname{atan}\left[\frac{\left(\sqrt{1+4\zeta^4} - 2\zeta^2\right)^{1/2}}{2\zeta}\right]$$

$\zeta = 0.5$; $PM = 51^\circ$

$\zeta = 0.6$; $PM = 59^\circ$

$\zeta = 0.7$; $PM = 65^\circ$

$\zeta = 1$ $PM = 76^\circ$

For most typical Systems

$$\boxed{PM = 100\zeta \text{ in degrees.}}$$

Time-domain Specs. and Bode plot.

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The time domain Specs. depend on ζ and ω_0

→ $t_p = \dfrac{\pi}{\omega_d}$; $\omega_d = \omega_0\sqrt{1-\zeta^2}$

→ $t_r = \dfrac{\pi - \phi}{\omega_d}$; $\phi = \cos^{-1}\zeta$.

→ $t_s = \dfrac{4}{\zeta\omega_0}$; 2% settling time

→ $M_p = e^{\left(-\pi\zeta/\sqrt{1-\zeta^2}\right)}$

→ $\omega_{gc} \cong \dfrac{\omega_0}{1.4}$; $PM \doteq 180 + \angle L(j\omega_{gc})$
$\cong 100\zeta$.

→ ω_{gc} and PM can be determined by the Bode plot; thus fixing ω_0 and ζ.

→ Note that the steady state error due to steps, ramps or parabolic inputs gets determined by the behavior of $L(j\omega)$ for ω close to zero (see next page).

→ ζ and ω_0 all get approximately determined by the Bode plot and thus determine time-domain properties.

Selection of type

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* Note that the steady state error in tracking steps is

$$\frac{1}{1+K_p}$$

where $K_p = G(0)\,K(0) = L(0)$

* Steady state error tracking ramps is

$$\frac{1}{K_v} \quad \text{where } K_v = \lim_{s \to 0} s\,G(s)\,K(s) = \lim_{s \to 0} s\,L(s)$$

* steady state error for tracking parabolic inputs is

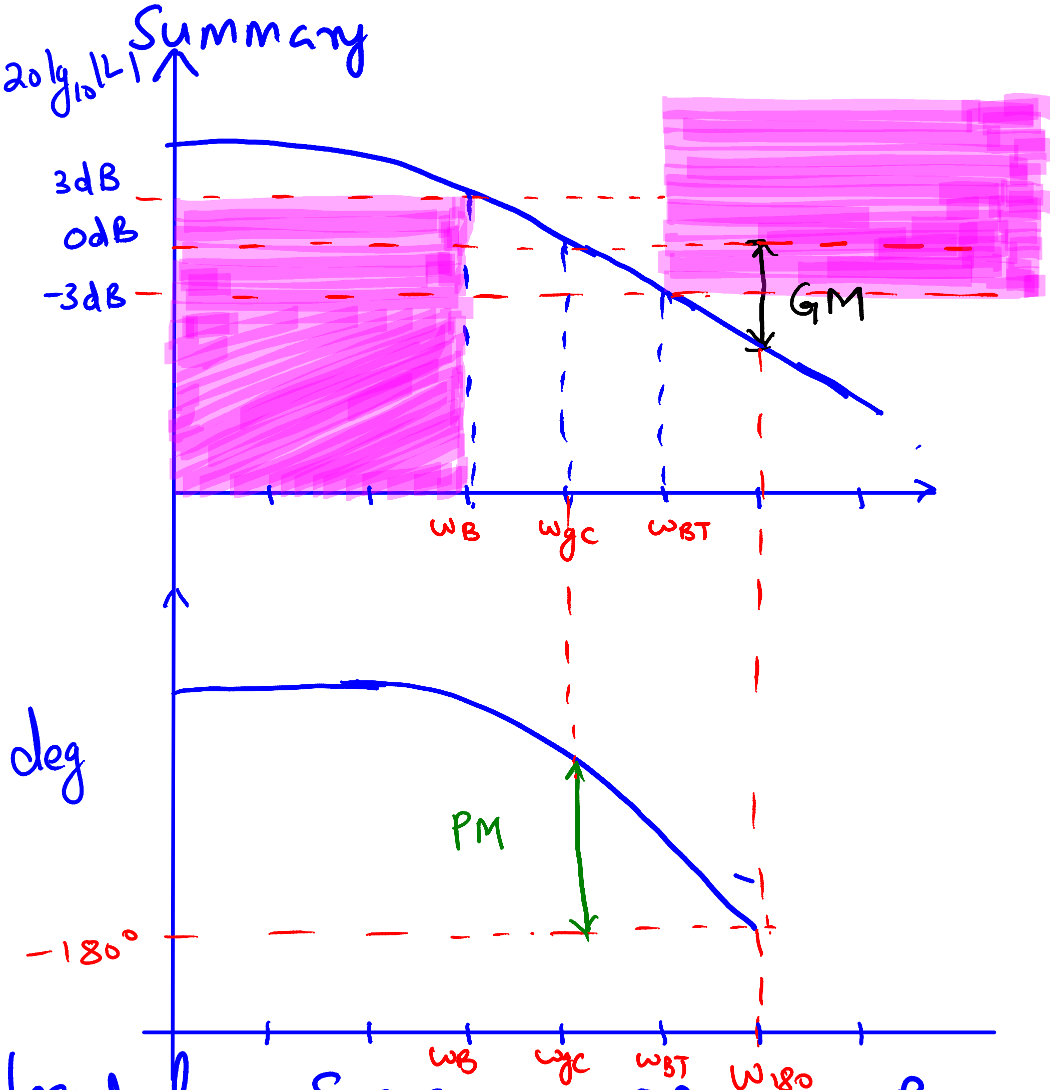
$$\frac{1}{K_a} \quad \text{where } K_a = \lim_{s \to 0} s^2\,G(s)\,K(s) = \lim_{s \to 0} s^2\,L(s)$$

* Therefore need −20dB/decade slope near DC for tracking steps (for the Bode plot)

* Need −40dB/decade slope near DC for tracking ramps

* Need −60dB/decade slope near DC for tracking parabolic input

Summary



Closed-loop Specs:	OPEN LOOP
* Bandwidth BW =	$(1.2-1.6)\omega_{gc}$
* PM (in degrees) $= 100\zeta$.	Can be read as shown above
* GM	Can be read as shown above.
* $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	$\zeta = (\text{PM in deg})/100$
* $t_s = 4/\zeta\omega_0$; $t_p = \pi/(\omega_0\sqrt{1-\zeta^2})$	$\omega_0 = 1.4\,\omega_{gc}$
* $t_r = (\pi - \cos^{-1}\zeta)/(\omega_0\sqrt{1-\zeta^2})$	
* Steady state behavior	determined by the behavior of $L(j\omega)$ near $\omega=0$.

Controller design

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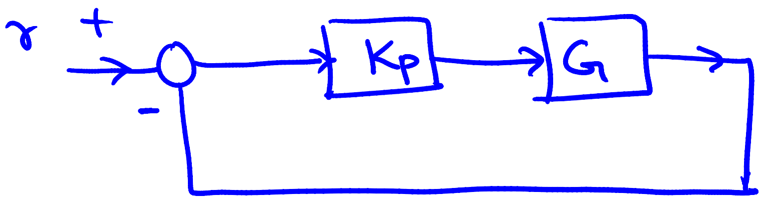
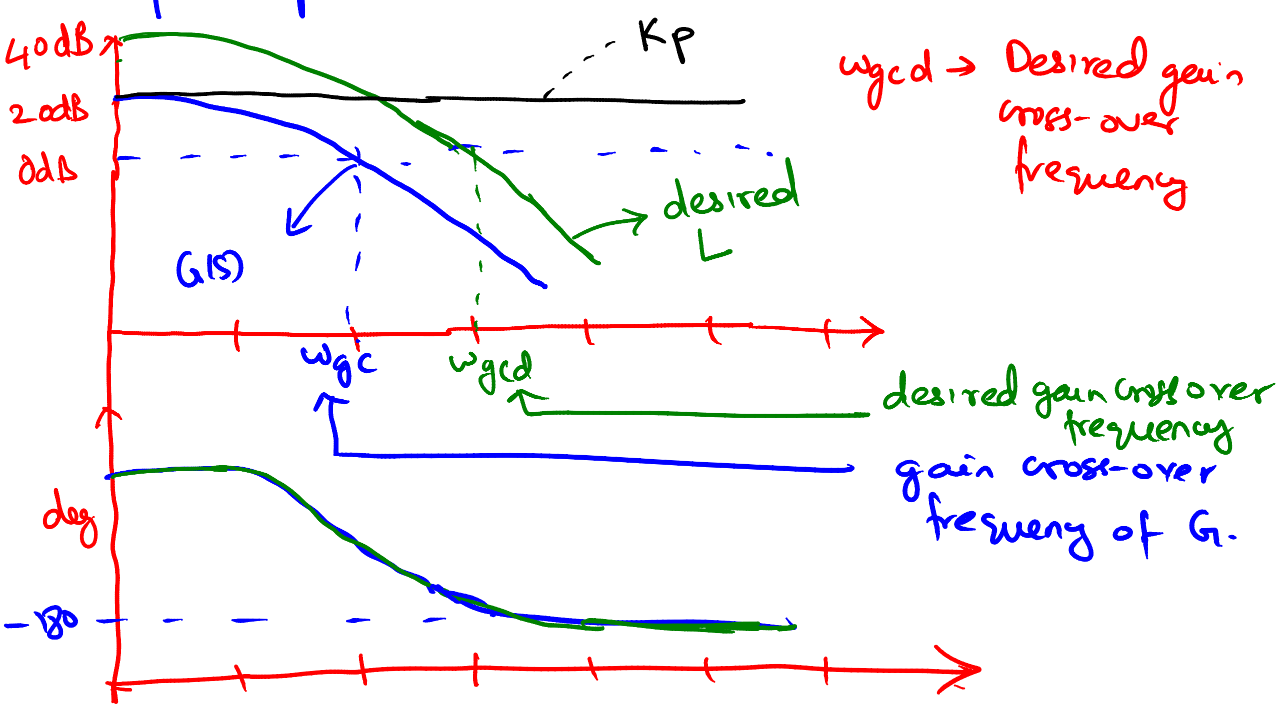
Proportional controller:



- $L = K_p G$.
- proportional controller $K = K_p$ a positive constant.

→ only one parameter to determine K_p.

Note that K_p can only move the magnitude Bode plot up or down without affecting the phase plot



Proportional controller.

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(1) Note that desired gain crossover frequency is ω_{gcd}

(2) To move $G(s)$ to desired L multiply by K_p i.e. $20\log_{10} K_p = 20\text{dB}$

$$\Rightarrow \log_{10} K_p = 1$$

$$\Rightarrow K_p = 10.$$

If the need is to have only one closed-loop characteristic chosen K_p can be similarly utilized

There is only one parameter to be designed!

→ Choose K to satisfy e_{ss} (steady state error)

→ Choose K to satisfy PM requirements

→ Choose K to satisfy t_r, t_s, M_p requirements.

Proportional integral controller

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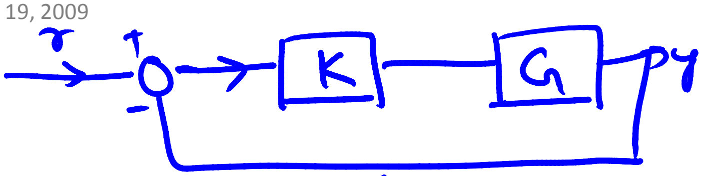


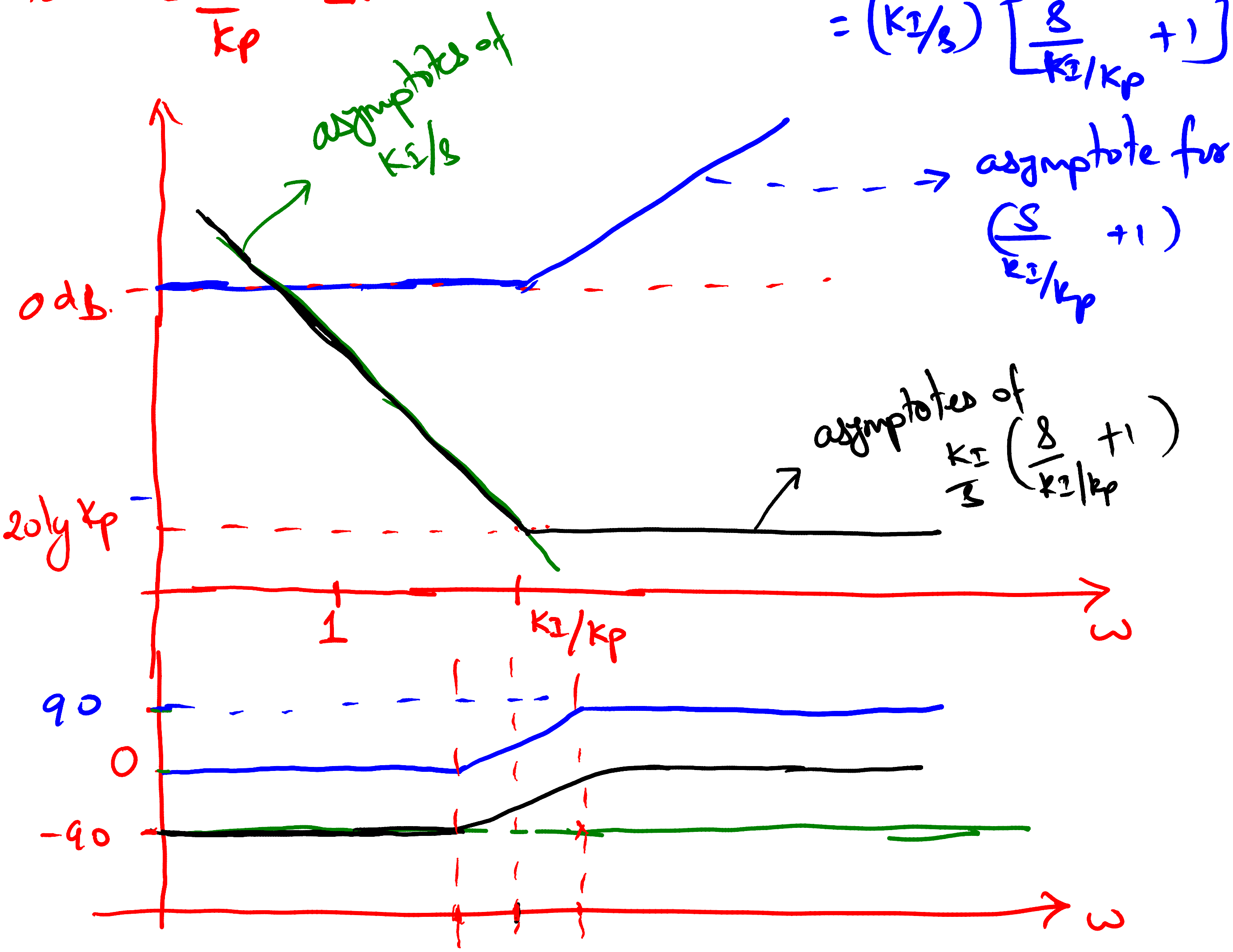
→ $K = K_p + \frac{K_I}{s} = \frac{sK_p + K_I}{s}$

* K_p is the proportional gain
* K_I is the integral gain

→ In the standard form $K(s) = \frac{K_I}{s}\left[\frac{sK_p}{K_I} + 1\right]$

$= (K_I/s)\left[\frac{s}{K_I/K_p} + 1\right]$

Assume $\frac{K_I}{K_p} > 1$.



Proportional - integral controller

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→ Note that the phase provided by a P-I controller is always negative.

- This will worsen the Phase-margin
- Phase ≈ 0 for $\omega \gg k_I/k_P$

→ The P-I controller has a slope of -20dB/decade in the dc region

- This will increase the type of the system.
  - ⇒ Suppose G is Type 0 then with PI controller L will have type 1

Thus the PI controller increases type of the interconnection

- Phase-margin can only be reduced by a PI controller; a destabilizing effect.

→ To reduce the destabilizing effect choose k_I/k_P such that $\omega_{gc} \gg k_I/k_P$ so that negative phase contribution due to PI controller is minimal effect on phase margin.

Phase Plot of PI controller.

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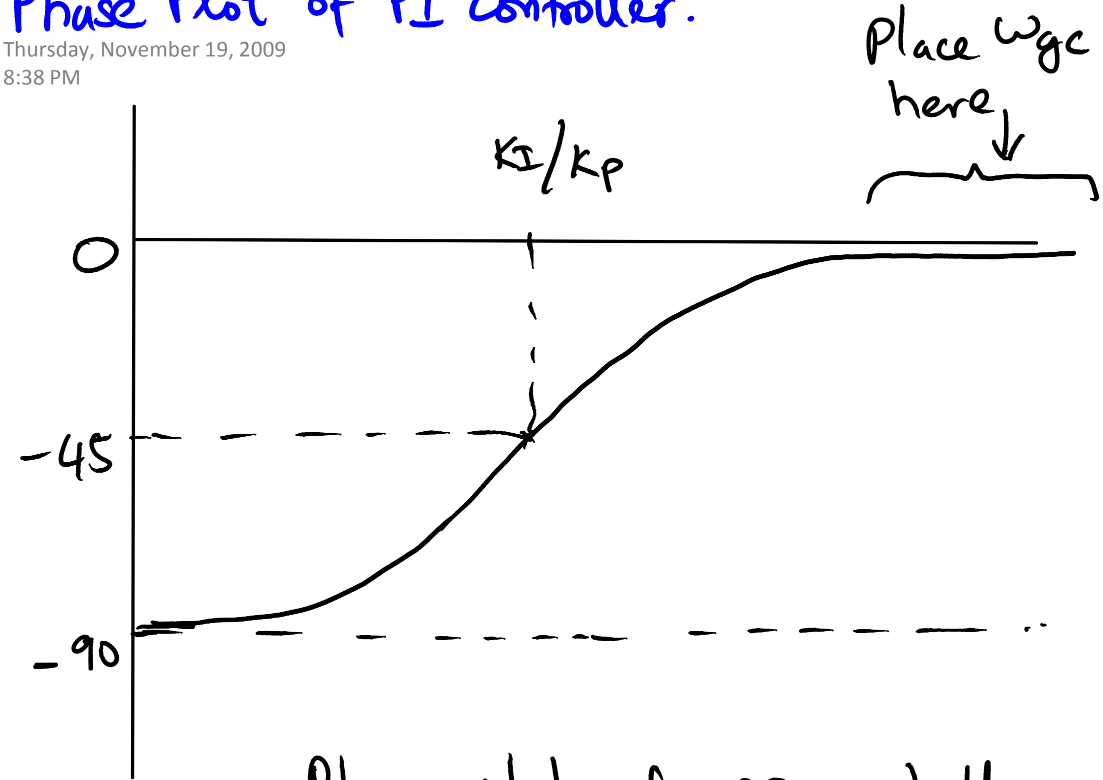

Phase-plot of PI controller.

→ choose $\frac{K_I}{K_p}$ such that $\omega_{gc} = \alpha K_I/K_p$

where $5 \leq \alpha \leq 10$.

Guidelines for designing PI controllers

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Design 1:

① Design K_p to fix M_p, phase-margin or ...

② Choose K_I such that

$$\frac{K_I}{K_p} \le \frac{\omega_{gcd}}{\alpha} \; ; \quad \alpha \text{ is between } 5 \text{ and } 10$$

and $\frac{K_I}{s} G(s)$ satisfy steady state error e_{ss} requirement.

Design 2:

① Select K_I such that

$$K_v = \lim_{s \to 0} \left[s \, \frac{K_I}{s} G(s) \right]$$

to meet steady state error due to ramps specification

② Note that the characteristic equation is

$$K_p G(s) + \frac{K_I}{s} G(s) + 1 = 0$$

$$\Rightarrow K_p \frac{G(s)}{\frac{K_I}{s} G(s) + 1} + 1 = 0$$

③ Do proportional design for $\frac{G(s)}{\frac{K_I}{s} G(s) + 1}$

PI design example

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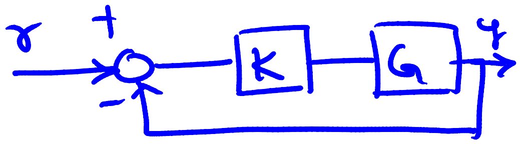

Suppose $G(s) = \dfrac{500}{s^2+6s+5}$

Design a PI controller to meet the following specifications on the closed-loop tf between r and y

(a) $M_p \le 16\%$

(b) e_{ss} (steady state error) due to ramp input ≤ 0.1

Solution:

Desired $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \le 0.16$

$$\Rightarrow \quad -\pi\xi/\sqrt{1-\xi^2} \le \ln 0.16.$$

$$\Rightarrow \quad \frac{\pi\xi}{\sqrt{1-\xi^2}} \geq -\ln 0.16$$

$$\Rightarrow \quad \frac{\pi\xi}{\sqrt{1-\xi^2}} \geq +1.8326$$

$$\Rightarrow \quad \xi \geq 0.5039.$$

$\therefore$ The Phase Margin desired

$PM_d = 100\xi \cong 50 + 7$ (Safety margin)

$= 57$ degrees.

Steady state error

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→ The transfer function between r and error e is

$$\frac{e}{r} = \frac{1}{1+L}$$

$$\therefore \quad e(s) = \frac{1}{1+L(s)} r(s)$$

when r is a ramp then; $e(s) = \frac{1}{1+L} \cdot \frac{1}{s^2}$

and the final value theorem states that

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s e(s)$$

$$= \lim_{s\to 0} \left[\frac{1}{1+L(s)} \cdot \frac{s}{s^2} \right]$$

$$= \lim_{s\to 0} \frac{1}{sL(s)+s} = \frac{1}{\lim_{s\to 0} sL(s)} = \frac{1}{K_v}$$

$\therefore$ For $e_{ss} \le 0.1$ we need

$$\frac{1}{K_v} \le 0.1 \Rightarrow K_v \geq 10.$$

$$K_v = \lim_{s\to 0} sL(s) = \lim_{s\to 0} s\left[K_p + \frac{K_I}{s}\right] G(s)$$

$$= \lim_{s\to 0} \left(s k_p L(s) + K_I L(s)\right) = K_I G(0)$$

[assuming stability of the interconnection)

$\therefore$ we need $K_I G(0) \geq 10.$

Example Continued.

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Thus, the specifications on the closed-loop translate to

(a) Phase Margin $\approx 57^\circ$

(b) $k_I G(0) \gtrsim 10.$

$\Rightarrow k_I 100 \gtrsim 10 \Rightarrow k_I \geq \frac{1}{10} = 0.1$

Bode of G

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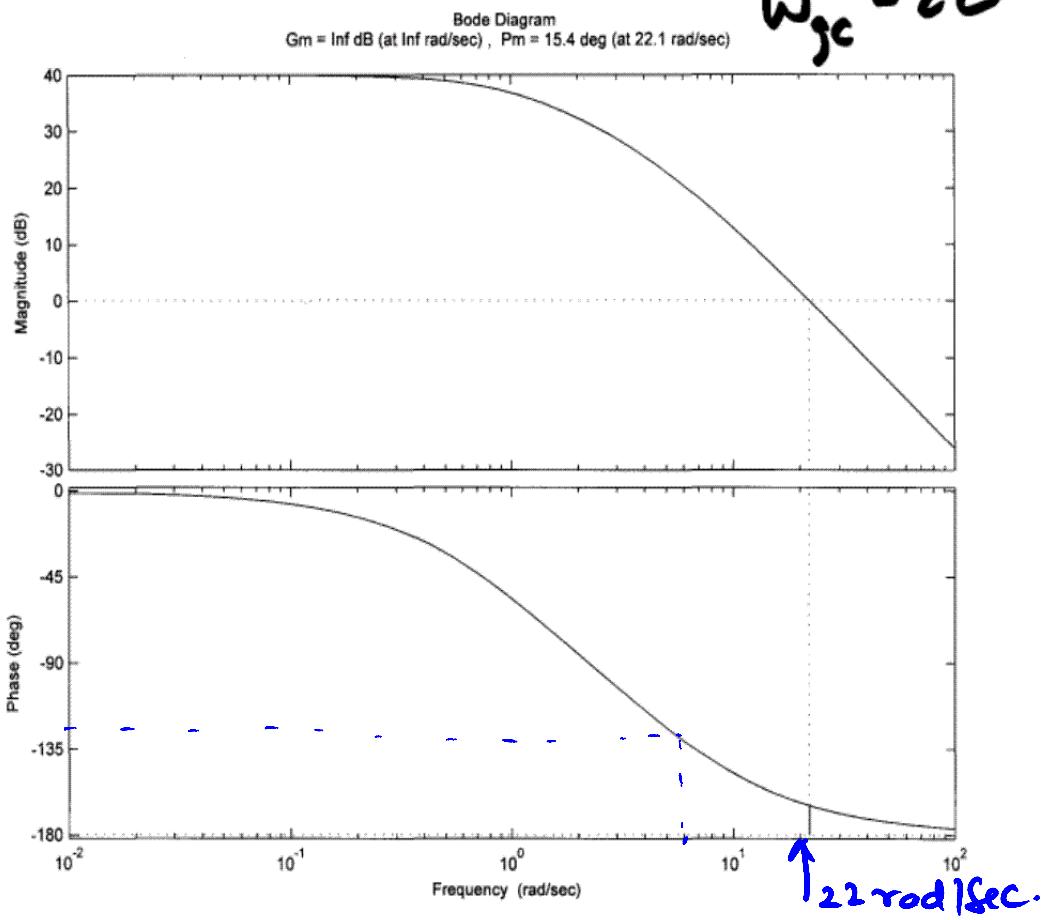
PM = 15°
ω_{gc} = 22



Comments: * The cross-over frequency with controller = 1; is ω_{gc} = 22 rad/sec.

* The associated phase Margin $PM_{existing}$ = 15°.

* Desired phase Margin PM_d = 57

∴ Extra Phase needed

* Note that at $\omega = 4.39$ $\angle G(j\omega) \cong -118$

∴ If gain crossover is moved to $\omega = 4.39$

PM will be 180 − 118 = 62 degrees.

Fixing K_p and K_I

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→ Choose K_p so that the gain crossover frequency of $L_1 = K_p G(s)$ is at

$\omega_{gcd} = 4.39$ rad/sec

$$\therefore \text{ Let } K_p = \frac{1}{|G(j4.39)|} \simeq 0.06$$

[This will result in $|L_1(j4.39)| = |K_p G(j4.39)| = 1$.]

→ Now choose K_I such that nothing gets effected near gain crossover frequency. This can be achieved by choosing $\frac{K_I}{K_p} = \frac{\omega_{gcd}}{\alpha}$; α in the range 5-10

$$= \left(\frac{4.39}{\alpha}\right) 0.06$$

→ Choosing $\alpha = 6$ we have $K_I = 0.044$

Note that we are not meeting $K_I > 0.1$ imposed by e_{ss} requirements.

Plots with k = 1

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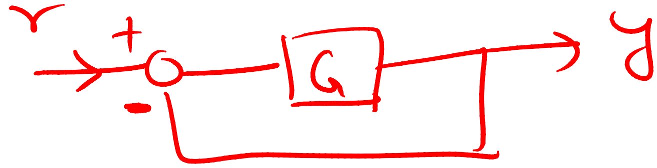
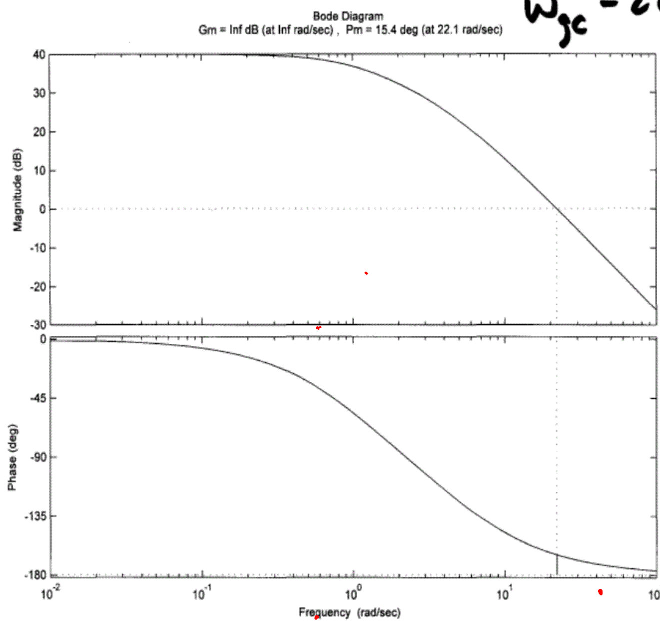


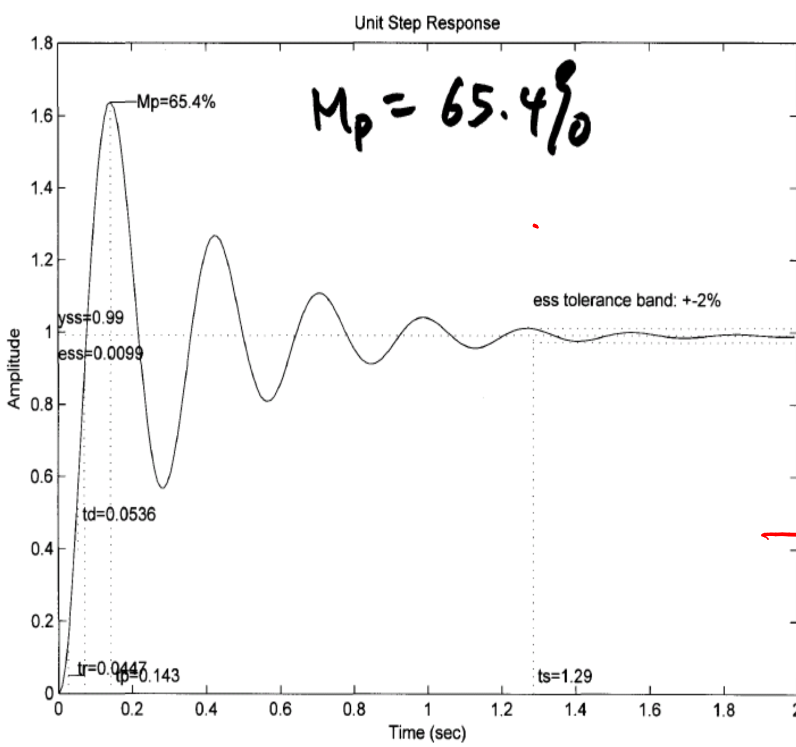
PM = 15°

$w_{gc} = 22$



Bode plot

$M_p = 65.4\%$



step response

Closed-loop characteristics with designed PI

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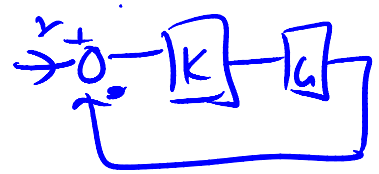
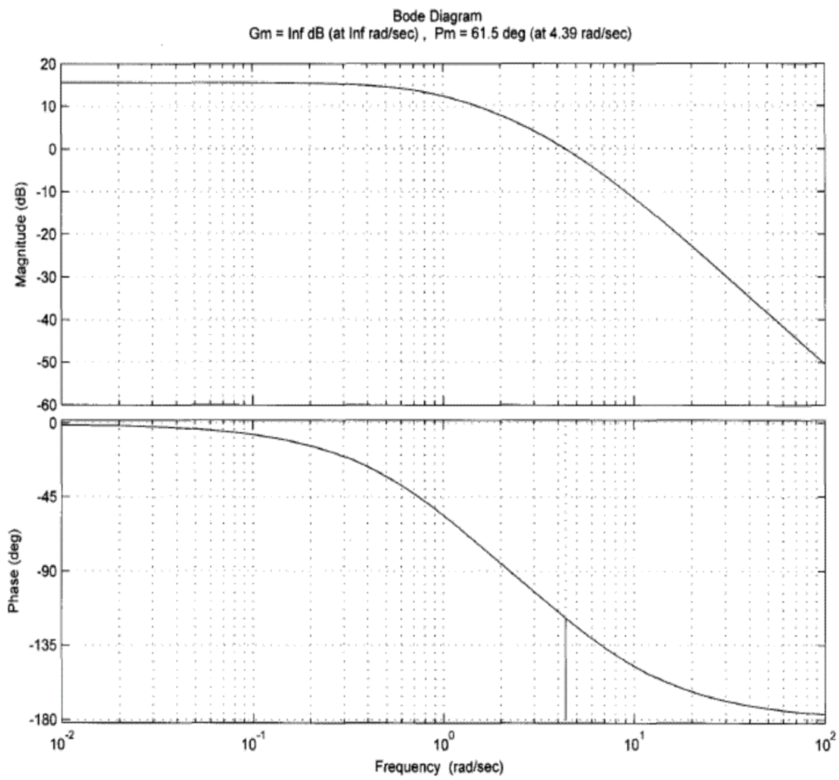
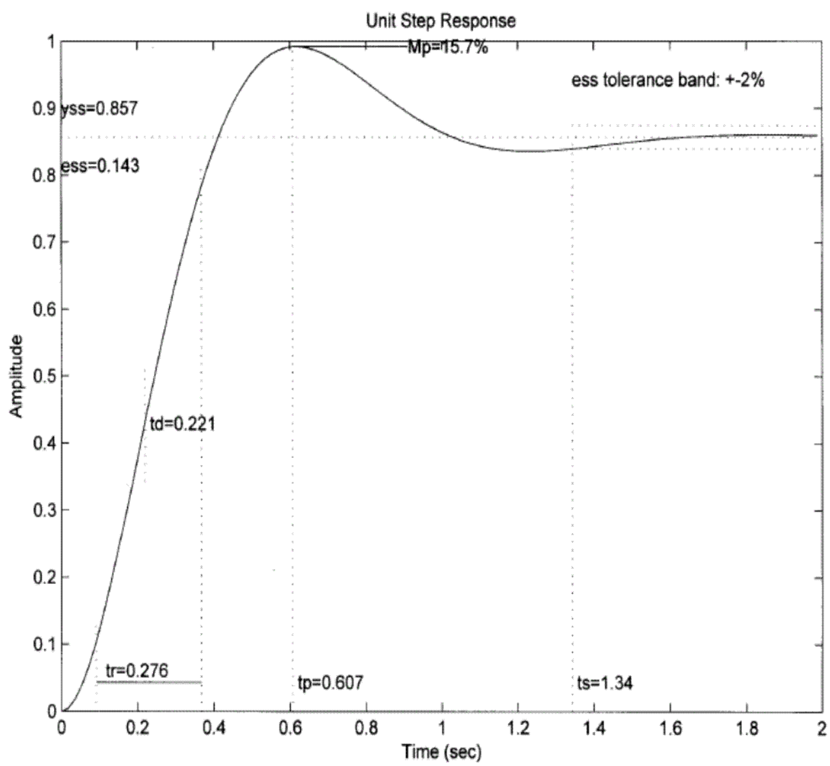


$$K = 0.06 + \frac{0.044}{s}$$





Note that

$K_v = 4.4$

instead of

10.

Iteration

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Note that $K_v \approx 4.4$ instead of 10

and therefore the e_{ss} requirement is not met.

→ There is some leeway for further iterating on K_p and K_I.
Note that desired overshoot is 0.16
and with the controller

$$0.06 + \frac{0.066}{s}$$

the overshoot is 0.15

Thus, there is "not much" leeway.