PARTIAL FRACTIONS

Wednesday, October 07, 2009
10:40 AM

Consider a polynomial

$d(s) = s^m + a_{m-1}s^{m-1} + \cdots + a_0$

(*) Then the above polynomial has m-zeros or m-roots. Let these roots be $s_1, s_2 \ldots s_m$. Some of these roots can be complex.

(*) Suppose the coefficients $a_0, a_1, \ldots, a_{m-1}$ are real numbers. Then, it follows that is $s_i = \alpha + j\beta$ is a root then $\bar{s}_i = \alpha_i - j\beta$ is also a root.

Proof: Suppose s_i is a root of $d(s)$

i.e. $d(s_i) = 0$

Then $s_i^m + a_{m-1}s_i^{m-1} + \cdots + a_0 = 0$

This implies

$\overline{d(s_i)} = 0$

$\Rightarrow \overline{s_i^m + a_{m-1}s_i^{m-1} + \cdots + a_0} = 0$

$\Rightarrow \overline{s_i^m} + \overline{a_{m-1}s_i^{m-1}} + \cdots + \bar{a}_0 = 0 \quad [\because \overline{x+y} = \bar{x} + \bar{y}]$

$\Rightarrow (\bar{s}_i)^m + (\bar{a}_{m-1})(\bar{s}_i)^{m-1} + \cdots + \bar{a}_0 = 0 \quad [\because \overline{xy} = \bar{x}\bar{y}]$

$\Rightarrow (\bar{s}_i)^m + a_{m-1}(\bar{s}_i)^{m-1} + \cdots + a_0 = 0 \quad [\because a_i \text{ are real } a_i = \bar{a}_i]$

$\Rightarrow d(\bar{s}_i) = 0$

$\therefore \bar{s}_i$ is also a root.

This completes the proof.

Complex Roots

Example:

$$G(s) = \frac{1}{s^2+s+1}$$

Roots of the polynomial as^2+bs+c where a, b, c are real are given by

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Thus the roots of s^2+s+1 are given by

$$s_1 = \frac{-1+\sqrt{1-4}}{2} \quad \text{and} \quad s_2 = \frac{-1-\sqrt{1-4}}{2}$$

$$\Rightarrow s_1 = \frac{-1+\sqrt{3}\,j}{2} \quad \text{and} \quad s_2 = \frac{-1-\sqrt{3}\,j}{2}$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}j \qquad\qquad = -\frac{1}{2} - \frac{\sqrt{3}}{2}j$$

Let $\alpha = -1/2$ and $\beta = \sqrt{\frac{3}{2}}$.

Thus,

$$G(s) = \frac{1}{s^2+s+1} = \frac{1}{(s-s_1)(s-s_2)}$$

$$\therefore G(s) = \frac{A}{s-s_1} + \frac{B}{s-s_2}$$

where $A = G(s)(s-s_1)\Big|_{s=s_1}$

$$= \frac{(s-s_1)}{(s-s_1)(s-s_2)}\Bigg|_{s=s_1} = \frac{1}{s-s_2}\Bigg|_{s=s_1}$$

$$= \frac{1}{s_1-s_2} = \frac{1}{\alpha+j\beta-(\alpha-j\beta)}$$

$$= \frac{1}{\alpha+j\beta-\alpha+j\beta}$$

$$= \frac{1}{2j\beta}$$

Similarly $\quad B = G(s)(s-s_2)\Big|_{s=s_2}$

$$= \frac{1}{s-s_1}\Big|_{s=s_2}$$

$$= \frac{1}{s_2-s_1} = -\frac{1}{2j\beta}$$

$$\therefore \quad G(s) = \frac{1}{2j\beta}\frac{1}{(s-s_1)} - \frac{1}{2j\beta}\frac{1}{(s-s_2)}$$

$$\therefore \quad g(t) = \frac{1}{2j\beta}\left[e^{s_1 t} - e^{s_2 t}\right]$$

$$= \frac{1}{2j\beta}\left[e^{(\alpha+j\beta)t} - e^{(\alpha-j\beta)t}\right]$$

$$= \frac{1}{2j\beta}\, e^{\alpha t}\left[e^{j\beta t} - e^{-j\beta t}\right]$$

$$= \frac{1}{\beta}e^{\alpha t}\left[\frac{e^{j\beta t} - e^{-j\beta t}}{2j}\right]$$

$$= \frac{1}{\beta}e^{\alpha t}\left(\sin\beta t\right) = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Another Method

We will first complete the squares

Indeed

$$s^2+s+1 = s^2+2\cdot\frac{1}{2}\cdot s+\frac{1}{4}-\frac{1}{4}+1$$

$$= \left(s+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$= \left(s+\frac{1}{2}\right)^2+\left(\sqrt{\frac{3}{4}}\right)^2$$

$$\therefore \quad G(s) = \frac{1}{s^2+s+1}$$

$$= \frac{1}{(s+\frac{1}{2})^2 + (\sqrt{\frac{3}{4}})^2}$$

$$= \frac{2}{\sqrt{3}} \quad \frac{\sqrt{3/4}}{(s+\frac{1}{2})^2 + (\sqrt{\frac{3}{4}})^2}$$

$$= \frac{1}{\beta} \quad \frac{\beta}{(s-\alpha)^2 + \beta^2}$$

Note that $\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2} =: F(s)$

and $\mathcal{L}\{e^{at}\sin\omega t\} = F(s-a) = \dfrac{\omega}{(s-a)^2+\omega^2}$

$$\therefore \quad \mathcal{L}^{-1} G(s) = \frac{1}{\beta} \mathcal{L}^{-1} \frac{\beta}{(s-\alpha)^2+\beta^2}$$

$$= \frac{1}{\beta} e^{\alpha t} \sin\beta t$$

$$= \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2})t$$

Frequency Response of LTI System

$$U(s) \longrightarrow \boxed{H(s)} \longrightarrow Y$$

We will assume $H(s)$ is stable.

$$U(s) = \frac{\omega}{s^2+\omega^2}$$

$$Y(s) = \frac{H(s)\,\omega}{s^2+\omega^2}$$

$$= \sum_{i=1}^{n} \frac{A_i}{s-s_i} + \frac{A}{s-j\omega} + \frac{B}{s+j\omega}$$

where $s_1, s_2, s_3, \dots, s_n$ are all poles of $H(s)$

where $s_1, s_2, s_3 \dots s_n$ are all poles of $H(s)$ with negative real parts.

$$A = Y(s)(s - j\omega)\Big|_{s=j\omega}$$

$$= \frac{H(s)\,\omega}{s^2+\omega^2}(s - j\omega)\Big|_{s=j\omega}$$

$$= \frac{H(j\omega)\,\omega}{s+j\omega} = \frac{H(j\omega)\omega}{2j\omega}$$

$$B = Y(s)(s+j\omega)\Big|_{s=-j\omega} = \frac{H(s)\,\omega}{s^2+\omega^2}(s+j\omega)\Big|_{s=-j\omega}$$

$$= \frac{H(s)\cdot\omega}{s-j\omega}\Big|_{s=-j\omega} = \frac{H(-j\omega)\,\omega}{-2j\omega}$$

$$\therefore Y(s) = \sum_{i=1}^{n} \frac{A_i}{s-s_i} + \frac{H(j\omega)\omega}{2j\omega}\frac{1}{s-j\omega} + \frac{H(-j\omega)\omega}{-2j\omega}\frac{1}{s+j\omega}$$

$$y(t) = \sum_{i=1}^{n} A_i e^{+s_i t} + \frac{H(j\omega)}{2j}e^{j\omega t} + \frac{H(-j\omega)}{-2j}e^{-j\omega t}$$

$$y_{ss}(t) = \left[\frac{H(j\omega)}{2j}e^{j\omega t} + \frac{H(-j\omega)}{-2j}e^{-j\omega t}\right]$$

$$= \frac{H(j\omega)e^{j\omega t}}{2j} + \overline{\frac{H(j\omega)e^{j\omega t}}{2j}}$$

$$= 2\,\text{Real}\left[\frac{H(j\omega)e^{j\omega t}}{2j}\right]$$

$$= \text{Real}\,\frac{|H(j\omega)|\,e^{j\angle H(j\omega)}e^{j\omega t}}{j}$$

$$= |H(j\omega)|\,\text{Re}\left[\frac{e^{j(\omega t+\angle H(j\omega))}}{j}\right]$$

$$= |H(j\omega)|\ \text{Re}\ \frac{1}{j}\left[\cos(\omega t+\angle H(j\omega))\right.$$

$$+ j \sin(\omega t + \angle H(j\omega))]$$

$$= |H(j\omega)| \sin(\omega t + \angle H(j\omega))$$

Thus a LTI system that is stable has a steady state response to $\sin \omega t$ given by

$$|H(j\omega)| \sin(\omega t + \angle H(j\omega))$$

where $H(s)$ is the Laplace transform of system's impulse response.

Note that an arbitrary sinusoid has the form

$$A \sin(\omega t + \theta)$$

The steady-state response of a stable LTI system to a sinusoid of the form above is

$$A |H(j\omega)| \sin(\omega t + \angle H(j\omega) + \theta).$$

Stability

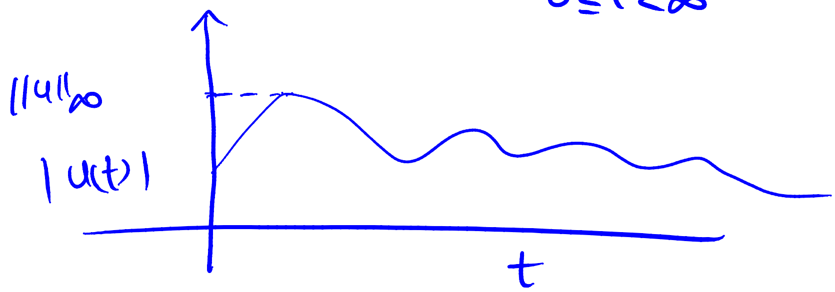
Friday, October 09, 2009
11:12 AM

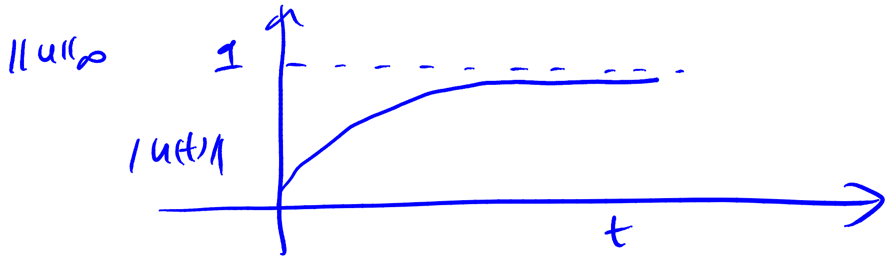
- Consider an input-output system S that is Linear-time-invariant and Causal.
- Suppose the impulse response of the system is given by $h(t); t \geq 0$.
- Then for an input $u(t)$ the output is $(Su)(t) = (h * u)(t)$

$$= \int_0^t u(\tau)\, h(t-\tau)\, d\tau$$

- Define the maximum magnitude of a signal by

$$\|u\|_\infty \doteq \max_{0 \leq t < \infty} |u(t)|$$





Theorem: System S is bounded-input bounded output stable if and only if

$$\int_0^\infty |h(t)|\,dt = M < \infty$$

Proof: (Sketch)

Suppose the input is bounded.

i.e. $\|u\|_\infty = \alpha < \infty$

Then the output y is given by

$$y(t) = \int_0^\infty h(t-\tau)\, u(\tau)\, d\tau$$

$$\Rightarrow |y(t)| = \left| \int_0^\infty h(t-\tau)\, u(\tau)\, d\tau \right|$$

$$\leq \int_0^\infty |h(t-\tau)|\, |u(\tau)|\, d\tau$$

$$\leq \int_0^\infty |h(t-\tau)| \max_{0 \leq \tau < \infty} |u(\tau)|\, d\tau$$

$$= \int_0^\infty |h(t-\tau)|\, \|u\|_\infty\, d\tau$$

$$= \|u\|_\infty \int_0^\infty |h(t-\tau)|\, d\tau$$

$$= \|u\|_\infty \int_0^\infty |h(\tau')|\, d\tau$$

$$\leq \alpha M =: \beta \qquad \text{for all } t.$$

[illegible]

Thus, $|y(t)| \leq \beta < \infty \ \forall \ t$

$\therefore$ $y(t)$ remains uniformly bounded by $\beta < \infty$.

$\therefore$ If the impulse response is absolutely integrable the system is BIBO (bounded-input-bounded-output) stable.

Suppose $\int_0^\infty |h(t)|\, dt = \infty$ ---- ⊛

Another way of stating ⊛ is that <u>given any constant M</u>, there exists a time T such that

$$\int_0^T |h(t)|\, dt \geq M.$$

Given any constant M; Let

$$u_T(\tau) = \mathrm{Sgn}\, h(T-\tau)$$

Then the output when the input is $u_T(\tau)$ is

$$y_T(T) = \int_0^\infty h(T-\tau)\, u_T(\tau)\, d\tau$$

$$= \int_0^\infty h(T-\tau)\, \mathrm{Sgn}(h(T-\tau))\, d\tau$$

$$= \int_0^\infty |h(T-\tau)|\, d\tau$$

$$\geq \int_0^T |h(t)|\, dt \geq M$$

$$\geq \int_0^T |h(t)|\, dt \geq M$$

$\therefore$ given any constant M, we can find $U_T(z)$ with $\max_{0 \leq z < \infty} |U_T(z)| = 1$

and the output magnitude at time T

$$|y_T(T)| \geq M$$

That means there is no constant β.

for which $\|u\|_\infty \leq 1 \Rightarrow \|y\|_\infty \leq \beta$

This completes the proof.

(*) We have seen that the output Laplace transform is given by

$$Y(s) = H(s)\, U(s)$$

and if

$$H(s) = E \frac{s^m + a_{m-1}s^{m-1} \cdots + a_0}{s^n + b_{n-1}s^{n-1} + \cdots + b_0}$$

and $d(s) = s^n + b_{n-1}s^{n-1} + \cdots + b_0$

has no zeros in the right half plane

then has the form

$$H(s) = \sum_i \frac{A_i}{(s-s_i)^{k_i}}$$

and as $Re(s_i) < 0$ it follows

and as $Re(s_i) < 0$ it follows that
$$h(t) = \sum_i t^{k_i} e^{+s_i t}$$

and $h(t)$ is absolutely integrable

This leads to the following theorem

Theorem: Suppose $h(t)$ is the impulse response of the system S and

$$\mathcal{L}\, h(t) = H(s) = E \, \frac{s^m + a_{m-1}s^{m-1} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} \cdots + b_0}$$

with all roots of the denominator polynomial in the strict RHP ($Re(s) < 0$) then $h(t)$ is absolutely integrable and the system is BIBO stable.

Routh Hurwitz

Friday, October 09, 2009
11:13 AM

* Definition: Transfer function

The Laplace transform of the impulse response of a LTI causal system is called the "transfer function" of the system.

* Suppose the transfer function of a system is given by

$$H(s) = E \frac{s^m + a_{m-1}s^{m-1} + \cdots + a_0}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

then $s_i; i = 1 \ldots n$ are the finite poles of the transfer function

* Thus, if the poles of the transfer function are in the L.H.P then system is BIBO stable

There are a couple of tests that allow for assessing whether the poles of a transfer function are in

Poles of a transfer function are in the L.H.P or not without explicitly evaluating the roots of the denominator. The first one we will develop is the Routh Hurwitz Criteria.

Routh Hurwitz:

Consider a polynomial of the form

$$F(s) = a_n s^n + a_{n-1} s^{n-1} \cdots + a_0 = 0$$

We will demonstrate the main concept by working on the polynomial

$$a_6 s^6 + a_5 s^5 + \cdots + a_0$$

The first row of the Routh table is formed by the leading coefficient followed by every alternate coefficient. Thus the first row is written as

$$s^6 \;:\; a_6 \quad a_4 \quad a_2 \quad a_0$$

indicates the leading power

Leading coefficient

coefficient of the power of s skipping s^5

skipping s^5

The second row is formed by staring with the coefficent of the second highest power of s and every other alternate term.

$s^5 \quad a_5 \quad a_3 \quad a_1 \quad 0$

Coefficient of second highest term.

Thus, the first two rows are given by

s^6	a_6	a_4	a_2	a_0
s^5	a_5	a_3	a_1	0
s^4	$\frac{a_5 a_4 - a_6 a_3}{a_5} = A$	$\frac{a_5 a_2 - a_6 a_1}{a_5} = B$	$\frac{a_5 a_0 - 0}{a_5} = a_0$	0
s^3	$\frac{A a_3 - a_5 B}{A} = C$	$\frac{A a_1 - a_5 a_0}{A} = D$	0	0
s^2	$\frac{CB - AD}{C} = E$	$\frac{C a_0}{C} = a_0$	0	0
s^1	$\frac{ED - C a_0}{E} = F$	0	0	0
s^0	$\frac{F a_0}{F} = a_0$	0	0	0

ROUTH CRITERION:

The roots of the polynomial are

in the strict left half plane if all the elements in the first column are of the same sign. The # of sign changes equals the # of roots in the RHP.

Example:

Consider $F(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$

(*) The leading power is 4; the first two rows and the table are

$$
\begin{array}{llll}
s^4 & 2 & 3 & 10 \\
s^3 & 1 & 5 & 0 \\
s^2 & \frac{(1)(3) - 5(2)}{1} = -7 & \frac{1(10) - 2(0)}{1} = 10 & 0 \\
s & \frac{(-7)(5) - (1)(10)}{-7} = 6.43 & 0 & 0 \\
s^0 & \frac{(6.43)(10)}{6.43} = 10 & 0 & 0
\end{array}
$$

Thus, the first column entries are

$$
\begin{pmatrix} 2 \\ 1 \\ -7 \\ 6.43 \\ 10 \end{pmatrix}
$$

where ~~are~~ there are two sign changes. Therefore, there are two roots in the RHP.

Special Cases:

(a) The first element in any row of the Routh's tabulation is zero but the whole row is not zero.

Example:

Consider $s^4 + s^3 + 2s^2 + 2s + 3$

The Routh table is

s^4	1	2	3
s^3	1	2	0
s^2	0	3	

There is a zero here. Cannot generate s^1 as we have to divide by zero

modify the table as follows and proceed

s^4	1	2	3
s^3	1	2	0
s^2	ε	3	0
s	$\frac{2(\varepsilon)-1(3)}{\varepsilon} \simeq -\frac{3}{\varepsilon}$	0	0
s^0	$\frac{-(3/\varepsilon)(3)}{-3/\varepsilon} = 3$	0	0

where the zero in the s^2 row is replaced by a small number $\varepsilon > 0$.

is replaced by a small number $\varepsilon > 0$.

Because there are two sign changes in
$$\begin{pmatrix} 1 \\ 1 \\ \varepsilon \\ -3/\varepsilon \\ 3 \end{pmatrix}$$
there are two roots in the RHP.

(*) An entire row in the Routh table is zero.

Example:

Consider the polynomial
$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4$$
The Routh table is

s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	0
s	0	0	0 ← zero row
s^0			

To obtain the next row form the auxiliary polynomial. from the previous row (in the example the s^2 row) that is nonzero

The auxiliary polynomial is given by

$$A(s) = 4s^2 + 4 \quad \text{coefficient of } s^0$$

differentiate the auxiliary polynomial

$$\frac{dA(s)}{ds} = 8s + 0$$

first coefficient of the 's' row

second coefficient of the 's' row

Thus, the Routh table is

s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	0
s	8	0	0
s^0	4	0	0

The first column is
$$\begin{pmatrix} 1 \\ 4 \\ 6 \\ 4 \\ 8 \\ 4 \end{pmatrix}$$

The conclusion is that there are

no roots in the region $\{ s \mid Re(s) > 0 \}$
but as an ~~extr~~ entire row was
zero the roots on the imaginary
axis are obtained by ~~sol~~ solving
for $A(s) = 0$ i.e.

$$4s^2 + 4 = 0$$

$\Rightarrow$ $s_1 = +j$ and $s_2 = -j$ are
two roots on the imaginary axis
that have $Re(s) = 0$.

The Routh Hurwitz Criterion is
best suited for selecting parameters
for stability. Consider a transfer
function whose denominator
polynomial is

$$s^3 + 3408.3s^2 + 1204000s + 1.5\times10^7 K.$$

The Routh table is given by

s^3	1	1204000
s^2	3408.3	$1.5\times10^7 K$
s	$\dfrac{(3408.3)(1204000) - 1.5\times10^7 K}{3408.3}$	0

So $1.5\times10^7 K$

The first column is given by

$$\begin{bmatrix} 1 \\ 3408.3 \\ 1204000 - \frac{1.5\times10^7}{3408.3}K \\ 1.5\times10^7 K \end{bmatrix}$$

For all elements in the above column to have the same sign we must have

$$1204000 - \frac{1.5\times10^7 K}{3408.3} > 0 \quad (1)$$

and

$$1.5\times10^7 K > 0 \longrightarrow (2)$$

from (1) we have.

$$K < \frac{(3408.3)(1204000)}{1.5\times10^7}$$

and from (2) $\quad K > 0$

$\therefore \quad K < 273.57$ and $K > 0$

Therefore for stability

$$0 < K < 273.57.$$

Interconnections

Monday, October 12, 2009
11:54 AM

* We have seen that a Linear time-invariant and Causal System LTIC with input u has an output

$$y(t) = (h * u)(t) = \int_0^t h(t-\tau)\,u(\tau)\,d\tau$$

where h is the impulse response of the system.

* We have seen that

$$\underset{u \neq 0}{\text{"max"}} \frac{\|y\|_\infty}{\|u\|_\infty} =: M < \infty$$

(which is also the definition of bounded-input bounded (BIBO) stable) if and only if the impulse response is absolutely integrable i.e.

$$\int_0^\infty |h(t)|\,dt = N < \infty$$

* We have seen that with the Laplace transform of the impulse response (called the "transfer function") of the form

$$E\,\frac{s^m + a_{m-1}s^{m-1} + \dots + a_0}{s^n + b_{n-1}s^{n-1} + \dots + b_0}$$

$$s^n + b_{n-1}s^{n-1} + \dots + b_0$$

where $m \leq n$; a_i and b_i are real the system is BIBO stable if and only if all the roots of the denominator polynomial (also called the poles of the transfer function) are in the strict left half plane (i.e. they lie in the set

$$\{s \mid Re(s) < 0\}).$$

Now we study Bounded-input Bounded output stability of interconnections

We will assume throughout that when a real-rational transfer function $G(s)$ is given with

$$G(s) = \frac{n(s)}{d(s)} \quad \text{where}$$

$n(s)$ and $d(s)$ are polynomials in s, then $n(s)$ and $d(s)$ have no common factors. (i.e. they are coprime polynomials).

Examples:

Examples:

Consider

$$G(s) = \frac{s-1}{s^2+2s+1}$$

here $n(s) = (s-1)$

$d(s) = s^2+2s+1 = (s+1)^2$

Note that there are no common factors between $n(s)$ and $d(s)$ and therefore they are co-prime.

Example:

Consider

$$G(s) = \frac{(s-1)}{s^2-2s+1} \quad \text{-----} \quad (*)$$

Here $n(s) = (s-1)$

$d(s) = s^2-2s+1$

$= (s-1)^2$

$n(s)$ and $d(s)$ have a common factor in $(s-1)$. Thus, the representation $(*)$ is not a coprime representation.

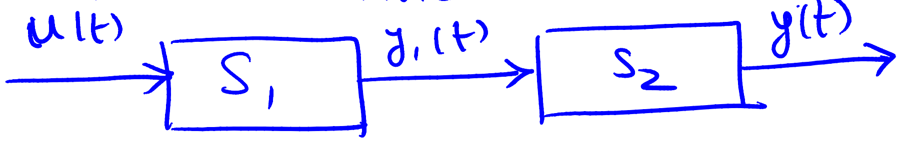
Note that

$G(s) = \frac{1}{(s-1)}$ which is a coprime representation.

a coprime representation.

Series Interconnection:

Consider two linear-time-invariant systems that are interconnected in a series manner.



Note that

$$y_1(t) = (h_1 * u)(t)$$

and $$y(t) = (h_2 * y_1)(t)$$

where $h_1(t)$ is the impulse response of S_1 and $h_2(t)$ is the impulse response of S_2.

Thus

$$y_1(t) = (h_2 * (h_1 * u))(t)$$

which is an unwieldly expression.

In the Laplace domain

$$\mathcal{L}(h * u)(t) = H(s)U(s)$$

where $H(s) = \mathcal{L}\, h(t)$ and

$$U(s) = \mathcal{L}\, u(t).$$

Lets consider the Bounded-input Bounded-output stability of the system.

System.

(*) Suppose S_1 is BIBO stable and S_2 is BIBO stable. Then

$H_1(s) = \frac{n_1(s)}{d_1(s)}$ (with $n_1(s)$ and $d_1(s)$ being co-prime) with no poles of $H_1(s)$ (which are the zeros of $d_1(s)$) in the rhp.

Similarly

$$H_2(s) = \frac{n_2(s)}{d_2(s)}$$ with no poles

in the rhp.

Then it follows that for the entire interconnection with $u(t)$ as the input and $y(t)$ as the output the transfer function is

$$H(s) = H_1(s)H_2(s) = \frac{n_1(s)\,n_2(s)}{d_1(s)\,d_2(s)}$$

Let $n'(s) = n_1(s)\,n_2(s)$

$d'(s) = d_1(s)\,d_2(s)$.

Its possible that $n'(s)$ and $d'(s)$ have common factors and therefore

$H(s) = \frac{n'(s)}{d'(s)}$ need not be

a coprime representation of $H(s)$

a coprime representation of $H(s)$ as $n_1(s)$ and $d_2(s)$ can have common factors and/or $n_2(s)$ and $d_1(s)$ can have common factors

Nevertheless, as

$H_1(s)$ and $H_2(s)$ are stable it follows that

$d_1(s)$ has no roots in the rhp and $d_2(s)$ also has no roots in the rhp. and

$$d_1(s) = (s-s_1) \cdots (s-s_m)$$

$$d_2(s) = (s-s_1') \cdots (s-s_p')$$

where $Re(s_i)<0$ and $Re(s_i')<0$

$$d'(s) = d_1(s)\, d_2(s)$$
$$= (s-s_1) \cdots (s-s_m)(s-s_1') \cdots (s-s_p')$$

Thus, even if some of the factors in $d'(s)$ are cancelled by the numerator $n'(s)$ the remaining poles will be in the rhp and thus, the transfer function $H(s)$ will be BIBO stable.

Summary: If $H_1(s)$ and $H_2(s)$ are both stable then $H(s) = H_1(s) H_2(s)$

are both stable then $H(s) = H_1(s)H_2(s)$ will be stable

Is it possible that $H(s)$ is BIBO stable with at least one of the systems S_1 or S_2 being BIBO unstable?

Answer: Yes; Consider

$$H_1(s) = \frac{s-1}{s+1}$$

and $$H_2(s) = \frac{1}{s^2-1}$$

Clearly, $H_1(s)$ is stable and

$$H_2(s) = \frac{1}{s^2-1} = \frac{1}{(s-1)(s+1)}$$ is

unstable; with poles 1 and -1.

Now the transfer function between input u and output y is given by

$$H(s) = H_1(s)\,H_2(s)$$

$$= \left(\frac{s-1}{s+1}\right)\frac{1}{s^2-1}$$

$$= \frac{\cancel{(s-1)}}{(s+1)}\,\frac{1}{\cancel{(s-1)}(s+1)}$$

$$= \frac{1}{(s+1)^2}$$

Note that $\frac{1}{(s+1)^2}$ is a coprime

Note that $\frac{1}{(s+1)^2}$ is a coprime representation and thus

$H(s)$ is a stable transfer function.

Thus,

$H(s)$ is stable (BIBO) but

$H_2(s)$ is not (BIBO) stable!

This is caused by "unstable pole-zero cancellation"; the unstable zero of the transfer function $H_1(s) = \frac{s-1}{s+1}$ cancels the unstable pole of the transfer function $H_2(s) = \frac{1}{(s-1)(s+1)}$.

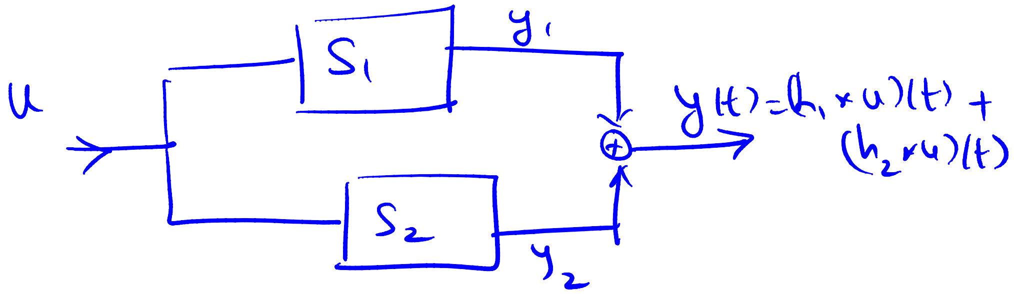
Thus, unstable pole-zero cancellations can hide instabilities in a system.

PARALLEL INTERCONNECTION:

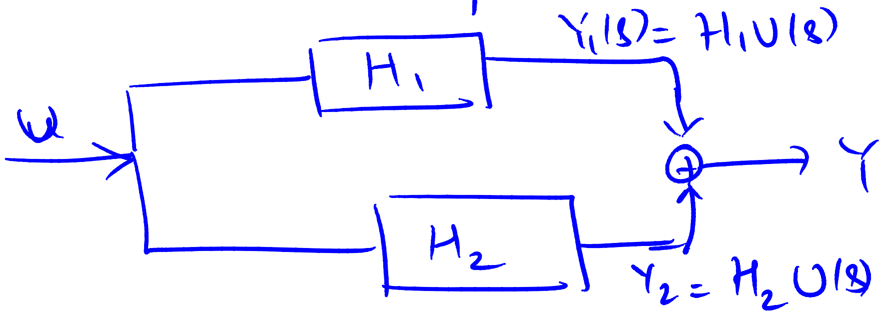
Consider two systems S_1 and S_2 with impulse responses h_1 and h_2 respectively and transfer functions $H_1(s)$ and $H_2(s)$ respectively.

Suppose, S_1 and S_2 are interconnected

Suppose, S_1 and S_2 are interconnected in a parallel architecture



In the transfer function domain



and $\quad Y(s) = H_1(s)\,U(s) + H_2(s)U(s)$

$$= \left[H_1(s) + H_2(s)\right] U(s).$$

Suppose $\quad H_1(s) = \dfrac{n_1(s)}{d_1(s)}$ and

$H_2(s) = \dfrac{n_2(s)}{d_2(s)}$ are two

coprime representations of H_1 and H_2 respectively

clearly if S_1 and S_2 are BIBO stable then $d_1(s)$ has no roots in the RHP and $d_2(s)$ has no roots in the RHP and therefore

$$H_1(s) + H_2(s) = \frac{n_1(s)}{d_1(s)} + \frac{n_2(s)}{d_2(s)}$$

$$= \frac{n_1 d_2 + d_1 n_2}{d_1(s)\, d_2(s)}$$

that has $d_1(s) d_2(s)$ as a denominator cannot have roots in the RHP.

Thus, if $H_1(s)$ and $H_2(s)$ are both BIBO stable then so is the transfer function obtained by $H_1(s) + H_2(s)$.

Thus, we have shown that

if $H_1(s)$ has no poles in RHP and $H_2(s)$ has no poles in RHP then

$H_1(s) + H_2(s)$ will have no poles in RHP

Question: Is it possible that $H_1(s) + H_2(s)$ has no poles in RHP but atleast one of $H_1(s)$ or $H_2(s)$ admits a pole in RHP?

Answer: Consider

$$H_1(s) = \frac{1}{s-1} + \frac{1}{s+1} = \frac{2s}{(s-1)^2}$$

$$H_2(s) = \frac{-1}{s-1} + \frac{1}{(s+3)} = \frac{-s-3+s-1}{(s-1)(s+3)} = \frac{-4}{(s-1)(s+3)}$$

Note that

$$H_1(s) + H_2(s) = \frac{2s}{(s-1)^2} - \frac{4}{(s-1)(s+3)}$$

$$= \frac{1}{s-1} + \frac{1}{s+1} - \frac{1}{s-1} + \frac{1}{s+3}$$

$$= \frac{1}{(s+1)} + \frac{1}{(s+3)}$$

$$= \frac{2s+4}{(s+1)(s+3)}$$

which is stable.

Thus, it is possible to have $H_1(s) + H_2(s)$ being unstable transfer function with both $H_1(s)$ and $H_2(s)$ being unstable transfer functions.

Is it possible to have $H_1 + H_2 = H(s)$ be a stable transfer function with only one of the transfer functions H_1 or H_2 being unstable?

Answer: No.

Suppose $H_1 + H_2 = H(s)$ is stable and suppose $H_2(s)$ is stable also

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then

$$H(s) = \frac{n(s)}{d(s)}$$

with $d(s)$ having no zeros in rhp.

$$H_2(s) = \frac{n_2(s)}{d_2(s)}$$ with $d_2(s)$

having no zeros in rhp.

Thus,

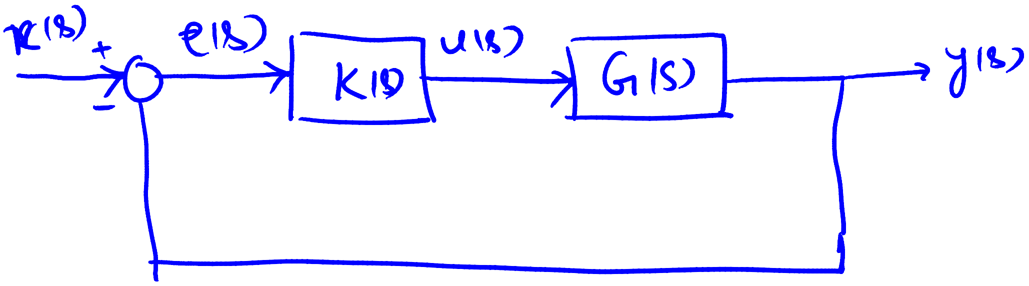
$$H_1(s) = H(s) - H_2(s)$$

$$= \frac{n(s)}{d(s)} - \frac{n_2(s)}{d_2(s)}$$

$$= \frac{n(s)d_2(s) - n_2(s)d(s)}{d(s)\,d_2(s)}$$

and as $d(s)d_2(s)$ have no zero in the rhp, $H_1(s)$ cannot have any poles in the rhp and thus $H_1(s)$ is also stable.

Unity Negative feedback interconnections



$$y(s) = G(s)\,u(s)$$

$$u(s) = K(s)\,e(s)$$

$$u(s) = K(s)\, e(s)$$

$$e(s) = r(s) - y(s)$$

$$\therefore \quad u(s) = K(s)\,[r(s) - y(s)]$$

$$\Rightarrow \quad y(s) = G(s)\,K(s)\,[r(s) - y(s)]$$

$$\Rightarrow \quad \big(1 + G(s)K(s)\big)\, y(s) = \underline{G(s)\,K(s)}\; r(s)$$

$$\Rightarrow \quad y(s) = \frac{G(s)\,K(s)}{1 + G(s)K(s)}\; r(s).$$

$\therefore$ The input $r(s)$ and output $y(s)$ transfer function is

$$\frac{G(s)\,K(s)}{1 + G(s)\,K(s)}$$

ⓧ Let $G = \dfrac{1}{(s-1)(s+1)}$ $\quad K(s) = \dfrac{s-1}{s+1}$

Then

$$\frac{G(s)\,K(s)}{1 + G(s)K(s)} = \frac{\dfrac{1}{(\cancel{s-1})(s+1)}\,\dfrac{\cancel{s-1}}{(s+1)}}{1 + \dfrac{1}{(\cancel{s-1})(s+1)}\,\dfrac{(\cancel{s-1})}{(s+1)}}$$

$$= \frac{\dfrac{1}{\cancel{(s+1)^2}}}{\dfrac{(s+1)^2 + 1}{\cancel{(s+1)^2}}}$$

$$= \frac{1}{1 + (s+1)^2}$$

which has poles at

$$1 + (s+1)^2 = 0$$

$$\Rightarrow (s+1)^2 = -1$$

$$\Rightarrow (s+1) = \pm j$$

$\therefore$ $s_1 = -1 + j$
$s_2 = -1 - j$ are the two poles

and $\dfrac{GK}{1+GK}$ is stable.