Robust Control: HW 7 and solutions

Problem 1: Find the poles and zeros of the following transfer matrix

$$\begin{bmatrix} \frac{s}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s+3} \\ \frac{-1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{1}{s} \end{bmatrix}$$

Solution: First we will describe how to obtain the Smith-McMillan form easily. Given a proper rational transfer matrix $G(s)$ with the $(ij)^{th}$ element being G_{ij}

1. first write

$$G(s) = \frac{1}{d(s)} P(s)$$

where $d(s)$ be the least common multiple of all the denominators of $G_{ij}(s)$.

For our example the least common multiple of the denominators is

$$d(s) = s(s+1)(s+2)(s+3)$$

and thus $P(s)$ is given by

$$P(s) = \begin{pmatrix} s^2(s+2)(s+3) & s(s+3) & s(s+1)(s+2) \\ -s(s+2)(s+3) & s(s+3) & (s+1)(s+2)(s+3) \end{pmatrix}.$$

2. Determine $\xi_i(s)$ the monic greatest common divisor of all the $i \times i$ minors of $P(s)$. Let $\xi_0(s) = 1$.

Note that for our example the 1×1 minors of $P(s)$ are all the individual elements of $P(s)$ given by $s^2(s+2)(s+3)$, $s(s+3)$, $s(s+1)(s+2)$, $-s(s+2)(s+3)$, $s(s+3)$, $(s+1)(s+2)(s+3)$. Thus

$$\begin{aligned} \xi_1(s) &= gcd\{s^2(s+2)(s+3), s(s+3), s(s+1)(s+2), -s(s+2)(s+3), (s+1)(s+2)(s+3)\} \\ &= 1 \end{aligned}$$

The 2×2 minors are given by

$$\begin{aligned} \xi_2(s) &= gcd\{s^2(s+3)^2(s+2)(s+1),\ s^2(s+1)(s+2)^2(s+3)(s+4),\ 3s(s+1)(s+2)(s+3) \\ &= s(s+1)(s+2)(s+3) \end{aligned}$$

3. Determine
$$\bar{\epsilon}_i(s) = \frac{\xi_i(s)}{\xi_{i-1}(s)}.$$

For the example

$$\begin{aligned} \xi_1(s) &= 1 \\ \xi_2(s) &= s(s+1)(s+2)(s+3) \end{aligned}$$

The Smith form is given by

$$\Sigma = \begin{pmatrix} \frac{\epsilon_1}{\psi_1} & & & 0 & \dots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \frac{\epsilon_r}{\psi_r} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Thus
$$G(s) = U\Sigma V$$

where $\frac{\epsilon_i}{\psi_i}$ is the coprime representation of $\frac{\bar{\epsilon}_i}{d(s)}$.

For our example

$$\begin{aligned} \frac{\epsilon_1}{\psi_1} &= \frac{1}{s(s+1)(s+2)(s+3)} \\ \frac{\epsilon_2}{\psi_2} &= \frac{s(s+1)(s+2)(s+3)}{s(s+1)(s+2)(s+3)} = 1 \end{aligned}$$

Thus the poles polynomial is given by $\psi_1(s)\psi_2(s) = s(s+1)(s+2)(s+3)$ and thus the poles are at $s = 0, -1, -2, -3$ and the zeros polynomial is $\epsilon_1(s)\epsilon_2(s) = 1$. Thus there are no zeros.

Problem 2: Prove that Suppose G_1 and G_2 have a state space realizations

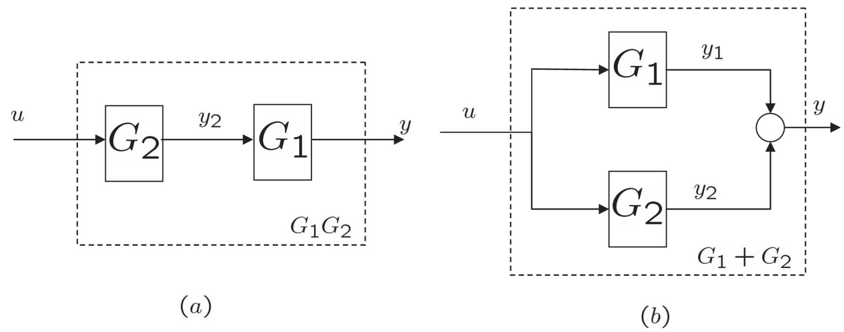

Figure 1:

$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ and $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ respectively. Then

- $$G_1G_2 = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right].$$

Solution: Refer to Figure 1. Let $y = G_1G_2u$. Let $G_2u = y_2$ then using the state space representation of G_2 and G_1 we have

$$\begin{aligned} \dot{x}_2 &= A_2x_2 + B_2u \\ y_2 &= C_2x_2 + D_2u \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + B_1y_2 = A_1x_1 + B_1C_2x_2 + B_1D_2u = A_1x_1 + B_1C_2x_2 + B_1D_2u \\ y &= C_1x_1 + D_1y_2 = C_1x_1 + D_1C_2x_2 + D_1D_2u \end{aligned}$$

Thus with $x = [x_1 \;\; x_2]^T$ we have

$$\dot{x} = \begin{pmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1D_2 \\ B_2 \end{pmatrix} u$$

and

$$y = [C_1 \;\; D_1C_2]x + D_1D_2u$$

The other realization is obtained by taking $x = [x_2 \;\; x_1]^T$.

- $$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

  Solution: Refer to Figure 1(b). Let $y = G_1 G_2 u$. Let $G_2 u = y_2$, $G_1 u = y_1$ and $y = y_1 + y_2$ then using the state space representation of G_2 and G_1 we have

  $$\begin{aligned} \dot{x}_2 &= A_2 x_2 + B_2 u \\ y_2 &= C_2 x_2 + D_2 u \end{aligned}$$

  $$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u \\ y &= y_1 + y_2 = C_1 x_1 + C_2 x_2 + (D_1 + D_2) u. \end{aligned}$$

  Thus with $x = [x_1 \;\; x_2]^T$ we have

  $$\dot{x} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$

  and

  $$y = [C_1 \;\; C_2] x + (D_1 + D_2) u$$

- Suppose $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is square and D is invertible then

  $$G^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right].$$

Solution: Let u be the input to G and y be its output. Then G^{-1} will map y (the input to G^{-1}) to u. G is described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y = &= Cx + Du \end{aligned}$$

$$\begin{aligned} u &= -D^{-1}Cx + D^{-1}y \\ \dot{x} &= Ax + B(D^{-1}y - D^{-1}Cx) = (A - BD^{-1}C)x + BD^{-1}y \end{aligned}$$

Problem 3: Prove that

$$\begin{pmatrix} I & -K \\ -G_{22} & I \end{pmatrix}^{-1} = \underbrace{\begin{pmatrix} (I-KG_{22})^{-1} & (I-KG_{22})^{-1}K \\ (I-G_{22}K)^{-1}G_{22} & (I-G_{22}K)^{-1} \end{pmatrix}}_{H(G_{22},K)}.$$

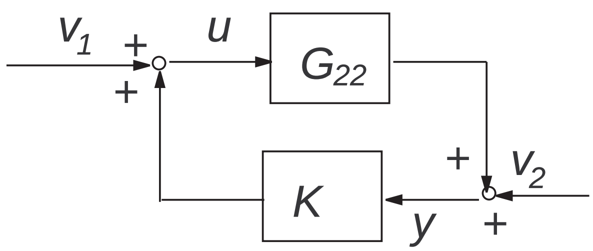

Figure 2:

Solution: Note that

$$\begin{pmatrix} I & -K \\ -G_{22} & I \end{pmatrix}\begin{pmatrix} (I-KG_{22})^{-1} & (I-KG_{22})^{-1}K \\ (I-G_{22}K)^{-1}G_{22} & (I-G_{22}K)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} (I-KG_{22})^{-1} - K(I-G_{22}K)^{-1}G_{22} & (I-KG_{22})^{-1}K - K(I-G_{22}K)^{-1} \\ -G_{22}(I-KG_{22})^{-1} + (I-G_{22}K)^{-1}G_{22} & -G_{22}(I-KG_{22})^{-1}K + (I-G_{22}K)^{-1} \end{pmatrix}$$

Note that

$$\begin{aligned} &(I-KG_{22})^{-1}K - K(I-G_{22}K)^{-1} \\ &= (I-KG_{22})^{-1}[K(I-G_{22}K) - (I-KG_{22})K](I-G_{22}K)^{-1} \\ &= (I-KG_{22})^{-1}[K - KG_{22}K - K + KG_{22}K](I-G_{22}K)^{-1} = 0 \end{aligned}$$

Therefore

$$(I-KG_{22})^{-1}K = K(I-G_{22}K)^{-1}.$$

$$\begin{array}{rcl} (I-KG_{22})^{-1} - K(I-G_{22}K)^{-1}G_{22} & = & (I-KG_{22})^{-1} - (I-KG_{22})^{-1}KG_{22} \\ & = & (I-KG_{22})^{-1}(I-KG_{22}) = I \end{array}.$$

Switching the roles of G_{22} and K one can prove that

$$-G_{22}(I-KG_{22})^{-1}+(I-G_{22}K)^{-1}G_{22} = 0, \quad -G_{22}(I-KG_{22})^{-1}K+(I-G_{22}K)^{-1} = I.$$

Problem 4: Consider Figure 2. Suppose G_{22} and K have minimal state space realizations $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22}\end{array}\right]$ and $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K\end{array}\right]$. Let

$$T = \left[\begin{array}{c|c} \overbrace{\begin{matrix} A & 0 \\ 0 & A_K \end{matrix}}^{A_1} & \overbrace{\begin{matrix} B_2 & 0 \\ 0 & B_K \end{matrix}}^{B} \\ \hline \underbrace{\begin{matrix} 0 & -C_K \\ -C_2 & 0 \end{matrix}}_{-C} & \underbrace{\begin{matrix} I & -D_K \\ -D_{22} & I \end{matrix}}_{D} \end{array}\right].$$

Thus

$$T^{-1} = \left[\begin{array}{c|c} A_1 + BD^{-1}C & BD^{-1} \\ \hline D^{-1}C & D^{-1}\end{array}\right] =: \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D}\end{array}\right]$$

where

$$\begin{aligned}
\overline{D} = D^{-1} &= \begin{pmatrix} I & -D_K \\ -D_{22} & I \end{pmatrix}^{-1} \\
&= \begin{pmatrix} I + (I - D_{22}D_K)^{-1}D_{22} & D_K(I - D_{22}D_K)^{-1} \\ (I - D_{22}D_K)^{-1}D_{22} & (I - D_{22}D_K)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} (I - D_{22}D_K)^{-1}D_{22} & D_K(I - D_{22}D_K)^{-1} \\ (I - D_{22}D_K)^{-1}D_{22} & (I - D_{22}D_K)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D_K \\ I \end{pmatrix} (I - D_{22}D_K)^{-1} \begin{pmatrix} D_{22} & I \end{pmatrix}
\end{aligned}$$

Thus

$$\overline{A} = A_1 + BD^{-1}C = \begin{pmatrix} A & B_2C_K \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix} (I - D_{22}D_K)^{-1} \begin{pmatrix} C_2 & D_{22}C_K \end{pmatrix}$$

Prove that the following are equivalent

1. $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ is stabilizable and detectable.
2. (A, B_2, C_2, D_{22}) and (A_K, B_K, C_K, D_K) are stabilizable and detectable.

Solution: (1) $\Rightarrow$ (2)

Suppose $(\overline{A}, \overline{C})$ is not detectable. Then there exists a $x = (x_G \;\; x_K)^T \neq 0$ partitioned according to the dimensions of A and A_K such that $\overline{A}x =$

λx with $\overline{C}x = 0$ with λ in rhp. This implies that $(A_1 + BD^{-1}C)x = 0$ and $D^{-1}Cx = 0$. Thus $A_1 x = 0 \text{with} Cx = 0$. Thus

$$\begin{pmatrix} A & 0 \\ 0 & A_K \end{pmatrix} \begin{pmatrix} x_G \\ x_k \end{pmatrix} = \lambda \begin{pmatrix} x_G \\ x_k \end{pmatrix}, \quad \begin{pmatrix} 0 & -C_K \\ -C_2 & 0 \end{pmatrix} \begin{pmatrix} x_G \\ x_k \end{pmatrix} = 0.$$

Thus

$$Ax_G = \lambda x_G, \quad C_2 x_G = 0, \quad A_K x_K = \lambda x_K, \; C_K x_K = 0.$$

As $x \neq 0$ atleast one of the vectors $x_G, \; x_K$ have to be nonzero. WLOG assume that $x_G \neq 0$. Then it follows that (A, C_2) is not detectable.

Suppose $(\overline{A}, \overline{B})$ is not stabilizable. Then there exists a $x^* = (x_G^* \; x_K^*) \neq 0$ such that $x^*\overline{A} = \lambda x^*, \quad x^*\overline{B} = 0$ with λ in rhp. This implies that $x^*(A_1 + BD^{-1}C) = \lambda x^*$ and $x^* B D^{-1} = 0$. Thus $x^* A_1 = 0$ and $x^* B = 0$. Thus

$$\begin{pmatrix} x_G^* & x_K^* \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A_K \end{pmatrix} = \lambda \begin{pmatrix} x_G^* & x_K^* \end{pmatrix}, \quad \begin{pmatrix} x_G^* & x_K^* \end{pmatrix} \begin{pmatrix} B_2 & 0 \\ 0 & B_K \end{pmatrix} = 0.$$

Thus

$$x_G^* A = \lambda x_G^*, \; x_G^* B_2 = 0, \; x_K^* A_K = \lambda x_K^*, \quad x_K^* B_K = 0.$$

As $x^* \neq 0$ at least one of the pairs $(A, B_2), (A_K, B_K)$ is not stabilizable.

Thus we have shown (1) $\Rightarrow$ (2). (2) $\Rightarrow$ (1) follows by similar line of argument.

Problem 5: Consider Figure 2.

1. Show that a realization of $L = G_{22}K$ is given by $\left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right]$ where

$$A_L = \begin{pmatrix} A & B_2C_K \\ 0 & A_K \end{pmatrix},\ B_L = \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix},\ C_L = (C_2\ \ D_{22}C_K),\ \ D_L = D_{22}D_K.$$

2. Show that a realization of

$$S = (I - L)^{-1} = \left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right]$$

where

$$\begin{aligned} A_S &= \overline{A} = \begin{pmatrix} A & B_2C_K \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix} (I - D_{22}D_K)^{-1} \begin{pmatrix} C_2 & D_{22}C_K \end{pmatrix} \\ B_S &= \begin{pmatrix} B_2D_K \\ B_K \end{pmatrix} (I - D_{22}D_K)^{-1} \\ C_S &= (I - D_{22}D_K)^{-1} \begin{pmatrix} C_2 & D_{22}C_K \end{pmatrix} \\ D_S &= (I - D_{22}D_K)^{-1} \end{aligned}$$

3. Prove that the following are equivalent:

   (a) (A_S, B_S, C_S, D_S) is stabilizable and detectable

   (b) (A_L, B_L, C_L, D_L) is stabilizable and detectable

Solutions: (1) from Problem 2.

(2) Note that $I - L = I - [C_L(sI - A_L)^{-1}B_L + D_L] = -C_L(sI - A_L)^{-1} + (I - D_L)$. Thus a realization of $I - L$ is given by $\left[\begin{array}{c|c} A_L & B_L \\ \hline -C_L & I - D_L \end{array}\right]$. Using Problem (2), a realization of $(I - L)^{-1}$ is given by $\left[\begin{array}{c|c} A_L + B_L(I - D_L)^{-1}C_L & B_L(I - D_L)^{-1} \\ \hline (I - D_L)^{-1}C_L & (I - D_L)^{-1} \end{array}\right]$. Substituting the realizations of L we obtain the result.

(3) Note that $A_S = A_L + B_L(I - D_L)^{-1}C_L,\ B_S = B_L(I - D_L)^{-1},\ C_S = (I - D_L)^{-1}C_L$ and $D_S = (I - D_L)^{-1}$. Thus it follows that

$$\begin{aligned} A_Sx = 0 \text{ and } C_Sx = 0 \quad &\Leftrightarrow \quad A_Lx = 0 \text{ and } C_Lx = 0 \\ z^*A_S = 0 \text{ and } z^*B_S = 0 \quad &\Leftrightarrow \quad z^*A_L = 0 \text{ and } z^*B_L = 0 \end{aligned}$$

The result follows from the above observation.

Problem 6: Prove that

1. If K is stable then the closed loop interconnection is stable if and only if $G_{22}(I - KG_{22})^{-1}$ is stable.
2. If G_{22} is stable then the closed loop interconnection is stable if and only if $K(I - G_{22}K)^{-1}$ is stable.

Solution: Note that the closed loop is internally stable if and only if

$$\begin{pmatrix} (I - KG_{22})^{-1} & (I - KG_{22})^{-1}K \\ (I - G_{22}K)^{-1}G_{22} & (I - G_{22}K)^{-1} \end{pmatrix}$$

is stable

(2) Note that in Problem (3) we have shown that

$$(I - KG_{22})^{-1}K = K(I - G_{22}K)^{-1}.$$

If the interconnection is stable then $(I - KG_{22})^{-1}K$ is stable and therefore $K(I - G_{22}K)^{-1}$ is stable.

If G_{22} is stable and $K(I - G_{22}K)^{-1}$ are stable then $(I - KG_{22})^{-1}K$ is stable. Note that $(I - KG_{22})^{-1} = I + (I - KG_{22})^{-1}KG_{22}$ which is also stable as G_{22} is stable. Note that

$$(I - G_{22}K)^{-1}G_{22} = G_{22}(I - KG_{22})^{-1}.$$

As both G_{22} and $(I - KG_{22})^{-1}$ are both stable $(I - G_{22}K)^{-1}G_{22}$ is stable. Finally $(I - G_{22}K)^{-1} = I + G_{22}(I - KG_{22})^{-1}K$ and therefore stable.

(1) can be proven by switching the roles of K and G_{22} in the proof above.