Homework 5

Robust Control

Problem 1 [Limitation due to rhp zero]

Consider the case of a plant with a rhp zero at z. The weight on the sensitivity transfer function is chosen as

$$w_p(s) = \frac{(\frac{1000s}{\omega_B} + \frac{1}{M})(\frac{s}{M\omega_B} + 1)}{(\frac{10s}{\omega_B} + 1)(\frac{100s}{\omega_B} + 1)}.$$

This weight is close to $1/M$ at low and high frequencies, has a maximum close to $10/M$ and intermediate frequencies and a asymptote that crosses 1 at frequencies $\omega_{BL}\omega_B/1000$ and $\omega_{BH} = \omega_B$. Thus we need good tracking ($|S| < 1$) in the frequency range between ω_{BL} and ω_{BH}.

1. Sketch $\frac{1}{|w_p|}$
2. Show that $|z|$ cannot be in the region where good tracking is needed and that we can achieve good tracking at frequencies either below $\frac{z}{2}$ or above $2z$. To see this select $M = 2$ and evaluate $w_p(z)$ at various values of $\omega_B = kz$, $k = 0.1, 0.5, 1, 10, 100, 1000, 2000$.

Problem 2 [Cart-Pendulum Problem]

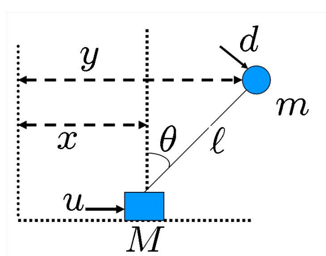

Figure 1: The inverted pendulum on a cart problem.

Consider Figure 1 that shows a inverted pendulum of length ℓ on a cart of mass M. The pendulum has a mass m on the opposite end of the cart. The position of mass m and the cart M are denoted by y and x respectively. The angle the pendulum makes with the vertical axis is given by θ. u and d are forces acting on the cart and the mass m respectively.

1. Show that the following equations of motion are satisfied

$$\begin{aligned}(M+m)\ddot{x} + m\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) &= u + d\cos\theta \\ m(\ddot{x}\cos\theta + \ell\ddot{\theta} - g\sin\theta) &= d\end{aligned}$$

2. Show that the equilibrium points of the dynamics are given by $(x, \theta, \dot{x}, \dot{\theta}) = (0,0,0,0)$ and $(x, \theta, \dot{x}, \dot{\theta}) = (0, \pi, 0, 0)$.

3. Obtain the linearized dynamics about the up position $((x, \theta, \dot{x}, \dot{\theta}) = (0,0,0,0))$ and the down position $((x, \theta, \dot{x}, \dot{\theta}) = (0, \pi, 0, 0))$.

4. Assess the stability of the equilibrium points.

5. Show that the transfer function from u to x and y in the up position are respectively

$$\frac{\ell s^2 - g}{D(s)} \text{ and } \frac{-g}{D(s)} \text{ with, } D(s) = s^2[M\ell s^2 - (M+m)g]$$

6. Evaluate the rhp poles and zeros of the up and the down position.

7. Consider the u to x transfer function for the up position.

   (a) Show tha the transfer function has a rhp pole p and zero z at

   $$p = z\sqrt{1+r} \text{ and } z = \sqrt{\frac{g}{l}}$$

   where $r = \frac{m}{M}$.

   (b) Show that as the ratio $r \to 0$ the stabilization problem becomes increasingly difficult. Show in particular that if $r = 0$ then its not possible to stabilize the dynamics by considering x as the measured variable and u as the control effort.

   (c) Comment on the ease of stabilization with respect to ℓ.

   (d) Assume multiplicative uncertainty and that robust stabilizability that is captured by the condition $\|W_I T\|_{\mathcal{H}_\infty} < 1$ is needed where W_I is a stable and proper weight with W_I being a high pass filter. Provide rationale why even if r is large, one cannot hope for good robustness. Show this aspect by illustrating that there is a inherent limitation on how small $\|W_I T\|_{\mathcal{H}_\infty}$ can be even when r is assumed large.

Thus the above exercise demonstrates that it is very difficult to stabilize a inverted pendulum using the position of the cart as the measured signal. Try an experiment of balancing a stick on your finger (you are the cart) by looking only at you finger. Let me know if you can do it.

8. Consider the transfer function between u and y and show that it does not suffer from the limitations seen in the transfer function between u and x.

   Thus the above exercise demonstrates that it is relatively easier to stabilize a inverted pendulum using the position of the farther end of the pendulum as the measured signal. Try an experiment of balancing a stick on your finger (you are the cart) by looking only at the end of the stick farther from your finger. You should be able to balance the stick. Let me know if you are not able to balance the stick.