Homework 4
Robust Control

Problem 1 Let $\mathcal{S}$ denote the set of all stable proper real-rational functions. Show that any element $G \in \mathcal{S}$ can be decomposed as $G = G_{ap}G_{mp}$ where G_{ap} and G_{mp} are all-pass and minimum phase respectively. Also prove that such a decomposition is unique upto a sign change.

Problem 2: Is it true that for every $\delta > 1$ there exists an internally stabilizing controller such that $\|T\|_{H_\infty} < \delta$.

Problem 3 Let W_p be the tracking performance weight that is chosen as a butterworth filter with cutoff frequency of 1 rad/s. Plot $|W_p(z)|$, $z = 0.1 + j\omega$, from $\omega = 0$ to a value of ω where $|W_p| < 0.01$. Repeat the same for abscissae of $z = 1 + j\omega$, and $z = 10 + j\omega$. Comment on the limitation imposed on the sensitivity transfer function S if the plant had zero at z.

Problem 4 Let $P = 4\frac{s-2}{(s+1)^2}$. Suppose K is an internally stabilizing controller in a unity negative feedback configuration such that $\|S\|_{H_\infty} = 1.5$. Give a positive lower bound for $\max_{0\leq\omega\leq 0.1} |S(j\omega)|$.

Problem 5 Define $\epsilon := \|W_pS\|_{H_\infty}$ and $\delta := \|KS\|_{H_\infty}$. As we have seen in class ϵ and δ capture tracking performance and controller effort respectively. Show that for every s_0 in the right half complex plane

$$|W_p(s_0)| \leq \epsilon + |W_p(s_0)P(s_0)|\delta.$$

This implies that ϵ and δ cannot be simultaneously made small indicating a tradeoff has to e reached between tracking and controller effort objectives.

Problem 6 Let ω be a frequency such that $j\omega$ is not a pole of the plant P in a unity negative feedback configuration. Suppose that $\epsilon := |S(j\omega)| < 1$. Derive a lower bound for $|K(j\omega)|$ where K is a stabilizing controller. Conclusion: *Good tracking at a particular frequency requires large controller gain at that frequency.*

Problem 7 Suppose that the plant is given by

$$P = \frac{1}{s^2 - s + 4}.$$

Suppose the controller K in a unity negative feedback configuration achieves the following

- Internal stability
- $|S(j\omega)| \leq \epsilon$ for $0 \leq \omega < 0.1$.
- $|S(j\omega)| \leq 2$ for $0.1 \leq \omega < 5$,

- $|S(j\omega)| \leq 1$ for $5 \leq \omega < \infty$

Find a positive lower bound on the achievable ϵ.

Problem 8

1. Prove that if a unity negative feedback system is stable, z_j, $j = 1, \dots, N_z$ are rhp zeros of the the plant, p_i, $i = 1, \dots, N_p$ are the rhp poles of the plant and θ denotes the time delay in the plant, then

$$M_S := \|S\|_{\mathcal{H}_\infty} \geq \Pi_{i=1}^{N_p} \frac{|z_j + p_i|}{|z_j - p_i|} =: M_{zp_i}, \quad \text{for all } j$$

and

$$M_T := \|T\|_{\mathcal{H}_\infty} \geq \Pi_{j=1}^{N_z} \frac{|z_j + p_i|}{|z_j - p_i|} |e^{p_i \theta}| =: M_{pz_j}, \quad \text{for all } i$$

with S and T denoting the sensitivity and complimentary sensitivity closed-loop transfer functions.

2. Prove that

$$\|S\|_{\mathcal{H}_\infty} \geq \|T\|_{\mathcal{H}_\infty} - 1.$$

Problem 9 Consider the plant

$$G(s) = 10 \frac{s - 2}{s^2 - 2s + 5}.$$

Show that $\|S\|_{\mathcal{H}_\infty} \geq 2.6$ and $\|T\|_{\mathcal{H}_\infty} \geq 2.6$.

Problem 10 The "minimum and stable version," G_{ms} of any transfer function G, is defined by

$$G_{ms} = \Pi_i \frac{s - p_i}{s + p_i} G(s) \Pi_j \frac{s + z_j}{s - z_j}.$$

We further denote

$$G_s(s) = \Pi_i \frac{s - p_i}{s + p_i} G(s) \text{ and } G_m(s) = G(s) \Pi_j \frac{s + z_j}{s - z_j}.$$

Let VT be a weighted complimentary sensitivity transfer function with V and T being the weight and complimentary sensitivity transfer function respectively.

Let V_{ms} be the "stable and minimum phase" version of V. Suppose p is a rhp pole of the plant G. Show that if the closed-loop systems is stable then

1. $\|VT\|_{\mathcal{H}_\infty} \geq |V_{ms}(p)|\Pi_{j=1}^{N_z}\frac{|z_j+p|}{|z_j-p|}|e^{p\theta}|$.

2. $\|KS\|_{\mathcal{H}_\infty} \geq |G_s(p)^{-1}$.

Problem 11 Consider the weight

$$w_p(s) = \frac{s+M\omega_B}{s}\frac{s+fM\omega_B}{s+fM^2\omega_B}$$

with $f > 1$. This weight is the same as the weight $\frac{s+M\omega_B}{s+\omega_B A}$ with $A = 0$ and that it approaches 1 at high frequencies where f gives a frequency range over which a peak is allowed. Plot the weight for $f = 10$ and $M = 2$. Derive and upper bound on ω_B in the case with $f = 10$ and $M = 2$.

Problem 12

Consider the weight

$$w_p = \frac{1}{M} + (\frac{\omega_B}{s})^n$$

on the sensitivity transfer function S. The weight requires the magnitude of the bode plot of $|S|$ to have a slope of n at low frequencies and requires its low frequency asymptote to cross 1 at the frequency ω_B. Derive an upper bound on ω_B when the plant has a rhp zero at z. Show that bound becomes smaller than $|z|$ as $n \to \infty$.

Problem 13 [Limitation due to rhp zero]

Consider the case of a plant with a rhp zero at z. The weight on the sensitivity transfer function is chosen as

$$w_p(s) = \frac{(\frac{1000s}{\omega_B} + \frac{1}{M})(\frac{s}{M\omega_B} + 1)}{(\frac{10s}{\omega_B} + 1)(\frac{100s}{\omega_B} + 1)}.$$

This weight is close to $1/M$ at low and high frequencies, has a maximum close to $10/M$ and intermediate frequencies and a asymptote that crosses 1 at frequencies $\omega_{BL}\omega_B/1000$ and $\omega_{BH} = \omega_B$. Thus we need good tracking ($|S| < 1$) in the frequency range between ω_{BL} and ω_{BH}.

1. Sketch $\frac{1}{|w_p|}$

2. Show that $|z|$ cannot be in the region where good tracking is needed and that we can achieve good tracking at frequencies either below $\frac{z}{2}$ or above $2z$. To see this select $M = 2$ and evaluate $w_p(z)$ at various values of $\omega_B = kz$, $k = 0.1, 0.5, 1, 10, 100, 1000, 2000$.