Computation of frequency responses of PDEs in Chebfun

Mihailo Jovanović
www.umn.edu/∼mihailo

joint work with:
Binh K. Lieu

University of Minnesota

2012 SIAM Annual Meeting

Example: heat equation

- **Distributed input and output fields**

$$\begin{aligned} \varphi_t(y,t) &= \varphi_{yy}(y,t) \, + \, d(y,t) \\ \varphi(y,0) &= 0 \\ \varphi(\pm 1,t) &= 0 \end{aligned}$$

⋆ **Harmonic forcing**

$$d(y,t) \,=\, d(y,\omega)\,\mathrm{e}^{\mathrm{j}\omega t} \quad \xrightarrow{\text{steady-state response}} \quad \varphi(y,t) \,=\, \varphi(y,\omega)\,\mathrm{e}^{\mathrm{j}\omega t}$$

⋆ **Frequency response operator**

$$\begin{aligned} \varphi(y,\omega) &= \left[\, \mathcal{T}(\omega)\, d(\,\cdot\,,\omega) \,\right](y) \\ &= \left[\, (\mathrm{j}\omega I \,-\, \partial_{yy})^{-1}\, d(\,\cdot\,,\omega) \,\right](y) \\ &= \int_{-1}^{1} T_{\mathrm{ker}}(y,\eta;\omega)\, d(\eta,\omega)\, \mathrm{d}\eta \end{aligned}$$

Two point boundary value realizations of $\mathcal{T}(\omega)$

- Input-output differential equation

$$\mathcal{T}(\omega) : \left\{ \begin{array}{rcl} \varphi''(y,\omega) \, - \, \mathrm{j}\omega \, \varphi(y,\omega) & = & -d(y,\omega) \\ \varphi(\pm 1,\omega) & = & 0 \end{array} \right.$$

- Spatial state-space realization

$$\mathcal{T}(\omega) : \left\{ \begin{array}{rcl} \left[\begin{array}{c} x_1'(y,\omega) \\ x_2'(y,\omega) \end{array} \right] & = & \left[\begin{array}{cc} 0 & 1 \\ \mathrm{j}\omega & 0 \end{array} \right] \left[\begin{array}{c} x_1(y,\omega) \\ x_2(y,\omega) \end{array} \right] + \left[\begin{array}{r} 0 \\ -1 \end{array} \right] d(y,\omega) \\ \varphi(y,\omega) & = & \left[\begin{array}{cc} 1 & 0 \end{array} \right] \left[\begin{array}{c} x_1(y,\omega) \\ x_2(y,\omega) \end{array} \right] \\ 0 & = & \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] \left[\begin{array}{c} x_1(-1,\omega) \\ x_2(-1,\omega) \end{array} \right] + \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} x_1(1,\omega) \\ x_2(1,\omega) \end{array} \right] \end{array} \right.$$

Frequency response operator

- Evolution equation

$$
\begin{aligned}
\left[\mathcal{E}\,\phi_t(\cdot,t)\right](y) &= \left[\mathcal{F}\,\phi(\cdot,t)\right](y) + \left[\mathcal{G}\,d(\cdot,t)\right](y), \quad y \in [a,\, b] \\
\varphi(y,t) &= \left[\mathcal{H}\,\phi(\cdot,t)\right](y), \quad t \in [0,\, \infty)
\end{aligned}
$$

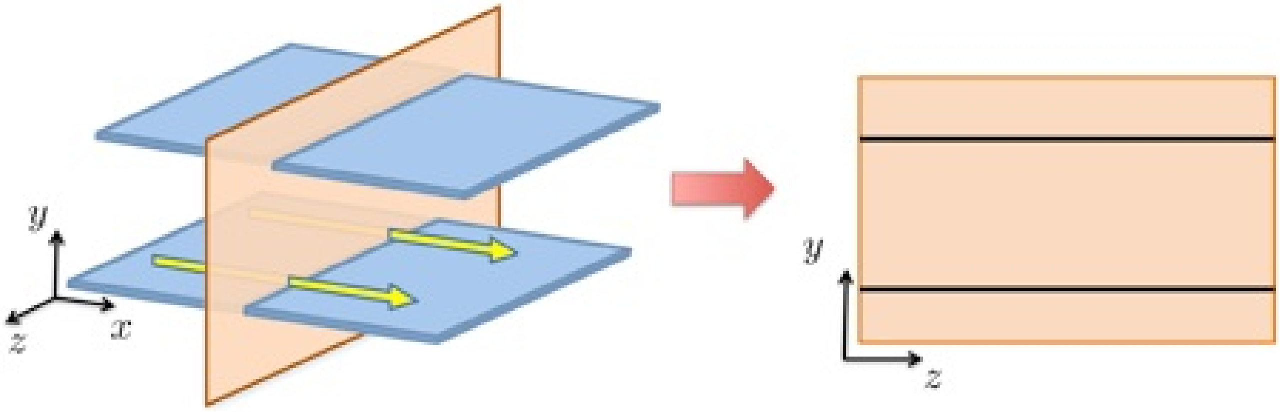
⋆ Spatial differential operators

$$
\mathcal{F} \;=\; [\mathcal{F}_{ij}] \;=\; \sum_{k\,=\,0}^{n_{ij}} f_{ij,k}(y)\,\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{d}y^k}
$$

⋆ Frequency response operator

$$
\mathcal{T} \;=\; \mathcal{H}\,\left(\mathrm{j}\,\omega\,\mathcal{E} \,-\, \mathcal{F}\right)^{-1}\mathcal{G}
$$

Example: channel flow



- **Streamwise-constant fluctuations**

$$\begin{bmatrix} \Delta & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \phi_{1t} \\ \phi_{2t} \end{bmatrix} = \begin{bmatrix} (1/Re)\,\Delta^2 & 0 \\ \mathcal{F}_{21} & (1/Re)\,\Delta \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Laplacian: $\Delta \;=\; \partial_{yy} \;-\; k_z^2$

"Square of Laplacian": $\Delta^2 \;=\; \partial_{yyyy} \;-\; 2\,k_z^2\,\partial_{yy} \;+\; k_z^4$

Coupling: $\mathcal{F}_{21} \;=\; -\,\mathrm{j} k_z\, U'(y)$

Singular value decomposition

- **Schmidt decomposition of a compact operator $\mathcal{T}(\omega)$: $\mathbb{H}_{\text{in}} \longrightarrow \mathbb{H}_{\text{out}}$**

$$\varphi(y,\omega) \;=\; [\mathcal{T}(\omega)\, d(\cdot,\omega)]\,(y) \;=\; \sum_{n=1}^{\infty} \sigma_n(\omega)\, u_n(y,\omega)\; \langle v_n, d\rangle$$

- **Left and right singular functions**

$$[\mathcal{T}(\omega)\, \mathcal{T}^\star(\omega)\, u_n(\cdot,\omega)]\,(y) \;=\; \sigma_n^2(\omega)\, u_n(y,\omega)$$

$$[\mathcal{T}^\star(\omega)\, \mathcal{T}(\omega)\, v_n(\cdot,\omega)]\,(y) \;=\; \sigma_n^2(\omega)\, v_n(y,\omega)$$

$\{u_n\}$ orthonormal basis of $\mathbb{H}_{\text{out}}$

$\{v_n\}$ orthonormal basis of $\mathbb{H}_{\text{in}}$

- **Right singular functions**
  - ⋆ **identify input directions with simple responses**

$$\sigma_1(\omega) \;\geq\; \sigma_2(\omega) \;\geq\; \cdots \;>\; 0$$

$$\varphi(\omega) \;=\; \mathcal{T}(\omega)\, d(\omega) \;=\; \sum_{n\,=\,1}^{\infty} \sigma_n(\omega)\, u_n(\omega)\, \langle v_n(\omega), d(\omega) \rangle$$

$$\Big\downarrow \; d(\omega) \;=\; v_m(\omega)$$

$$\varphi(\omega) \;=\; \sigma_m(\omega)\, u_m(\omega)$$

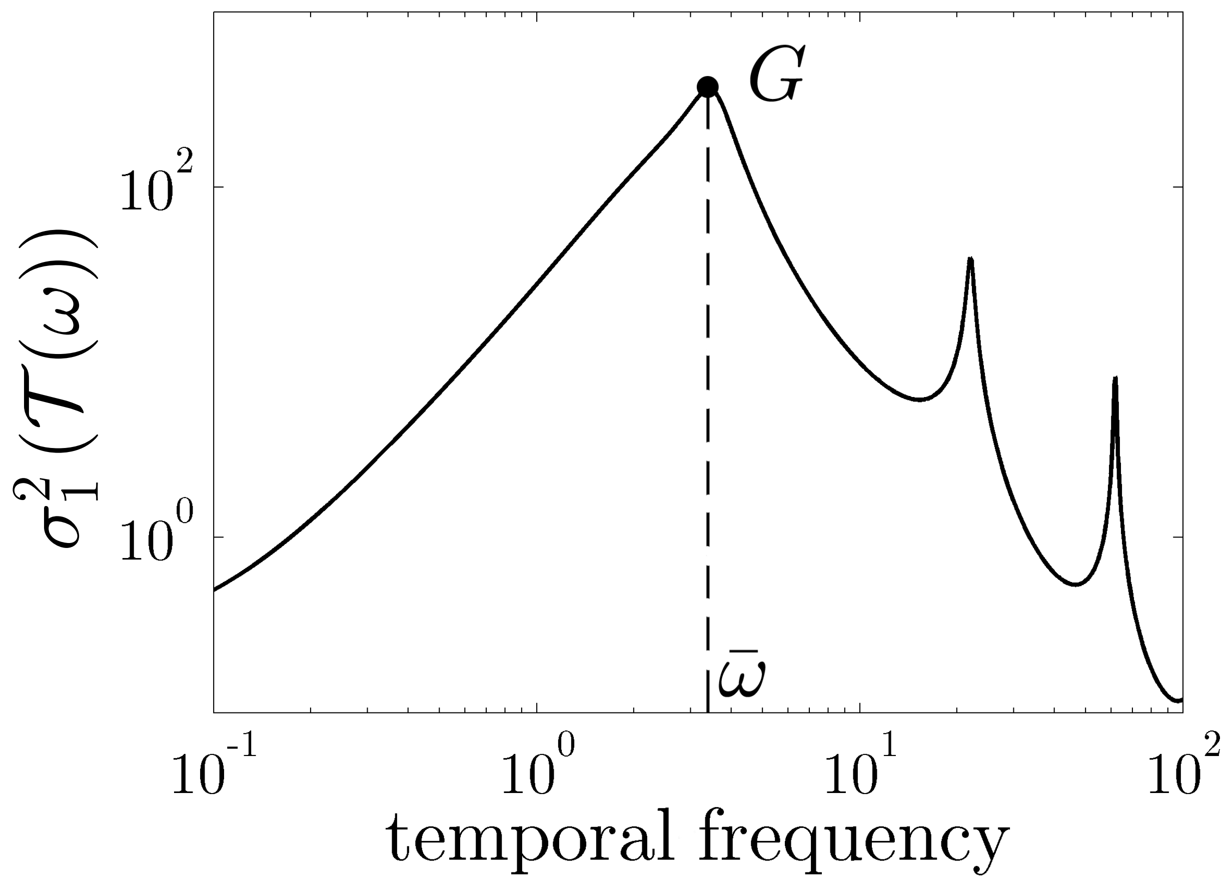
$\sigma_1(\omega)$**: the largest amplification at any frequency**

Worst case amplification

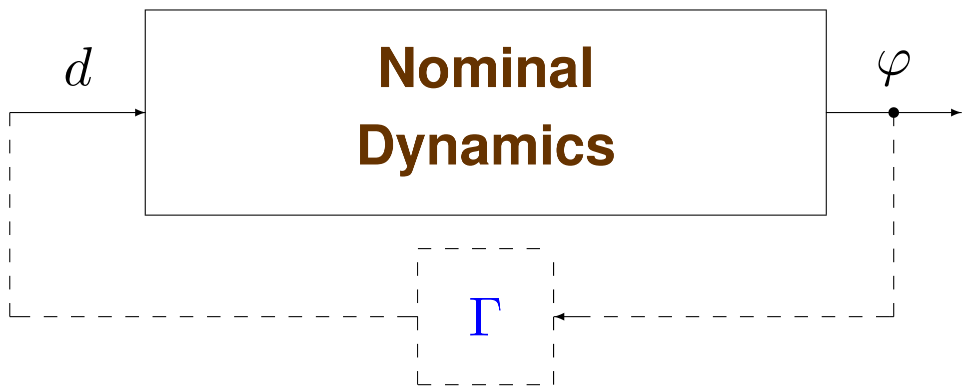
- H_∞ **norm: an induced L_2 gain (of a system)**

$$G \;=\; \sup \frac{\text{output energy}}{\text{input energy}} \;=\; \sup \frac{\displaystyle\int_0^\infty \langle \varphi(t), \varphi(t) \rangle \, \mathrm{d}t}{\displaystyle\int_0^\infty \langle d(t), d(t) \rangle \, \mathrm{d}t}$$

$$= \; \sup_\omega \sigma_1^2 \left(\mathcal{T}(\omega) \right)$$



Robustness interpretation



modeling uncertainty

(can be nonlinear or time-varying)

small-gain theorem:

stability for all Γ with $\max\limits_{\omega} \sigma_1^2(\Gamma(\omega)) \leq \gamma^2$	$\Leftrightarrow$	$\gamma^2 < \dfrac{1}{G}$

LARGE **worst case amplification** $\Longrightarrow$ **small stability margins**

closely related to pseudospectra of linear operators

Pseudo-spectral methods

- MATLAB *Differentiation Matrix Suite*

$$\mathcal{T}(\omega) \;=\; \left(\mathrm{j}\omega I \,-\, D^{(2)}\right)^{-1}$$

$\approx$  N

N

- Advantages
  - ⋆ superior accuracy compared to finite difference methods
  - ⋆ ease-to-use MATLAB codes

- Disadvantages
  - ⋆ ill-conditioning of high-order differentiation matrices
  - ⋆ implementation of boundary conditions may be non-trivial

Weideman & Reddy, ACM. TOMS. '00

Alternative method

1. Frequency response operator: two-point boundary value problem

2. Integral form of differential equations

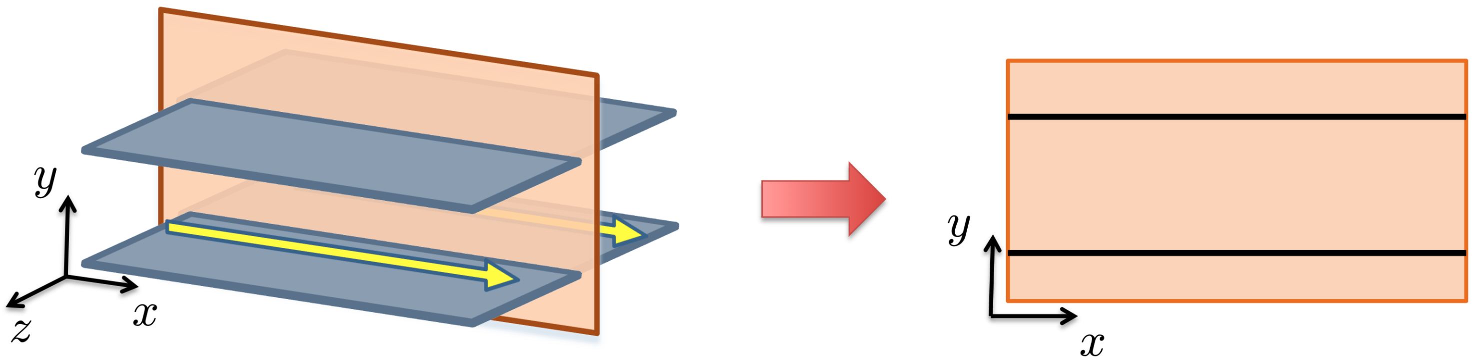
3. State-of-the-art automatic spectral collocation techniques

Advantages of Chebfun

- Superior accuracy compared to currently available schemes
- Avoids ill-conditioning of high-order differentiation matrices
- Incorporates a wide range of boundary conditions
- Easy-to-use MATLAB codes

Lieu & Jovanović
"Computation of frequency responses of linear time-invariant PDEs on a compact interval", submitted to J. Comput. Phys., 2011
Also arXiv:1112.0579v1

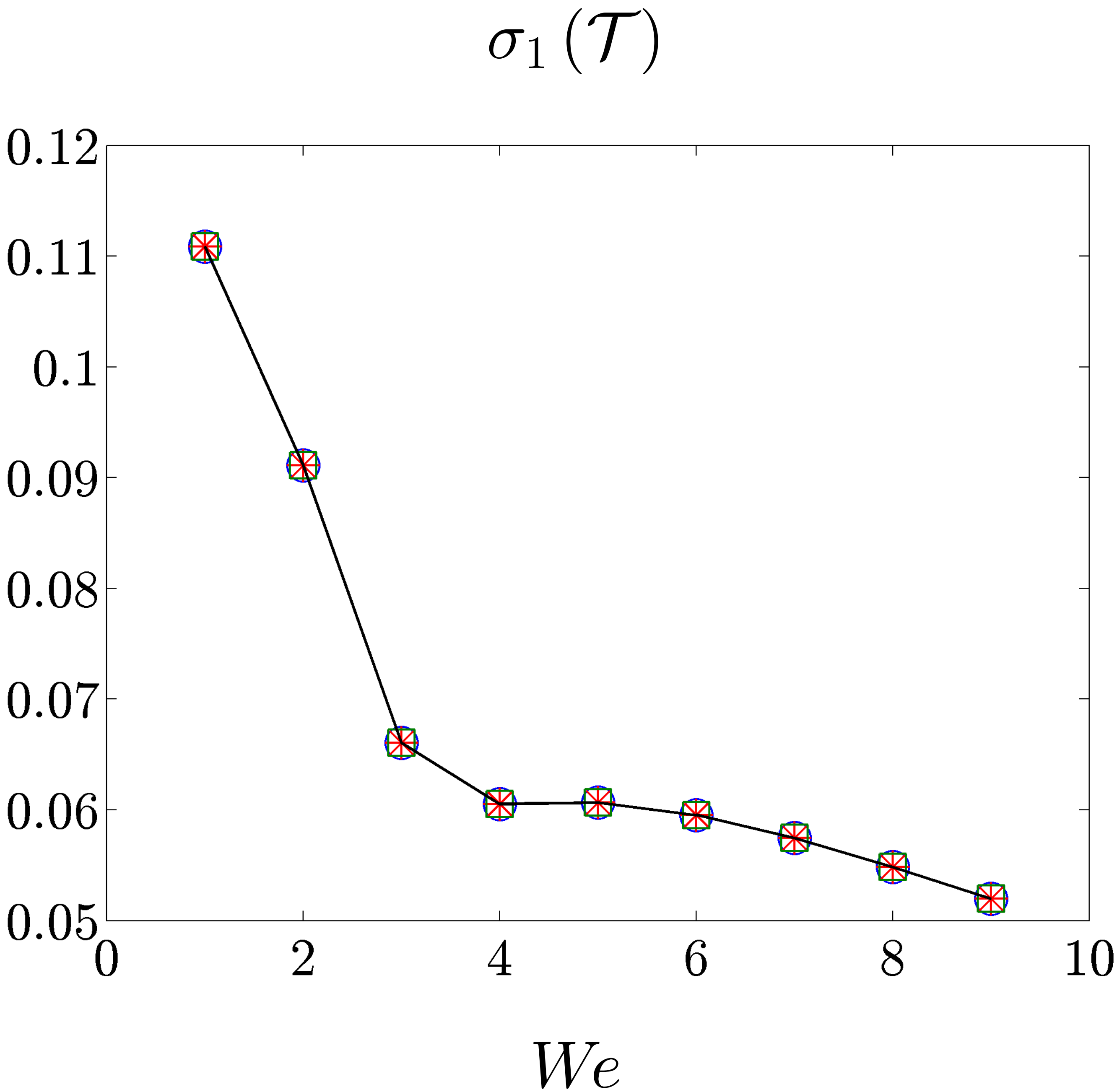
2D inertialess flow of viscoelastic fluids



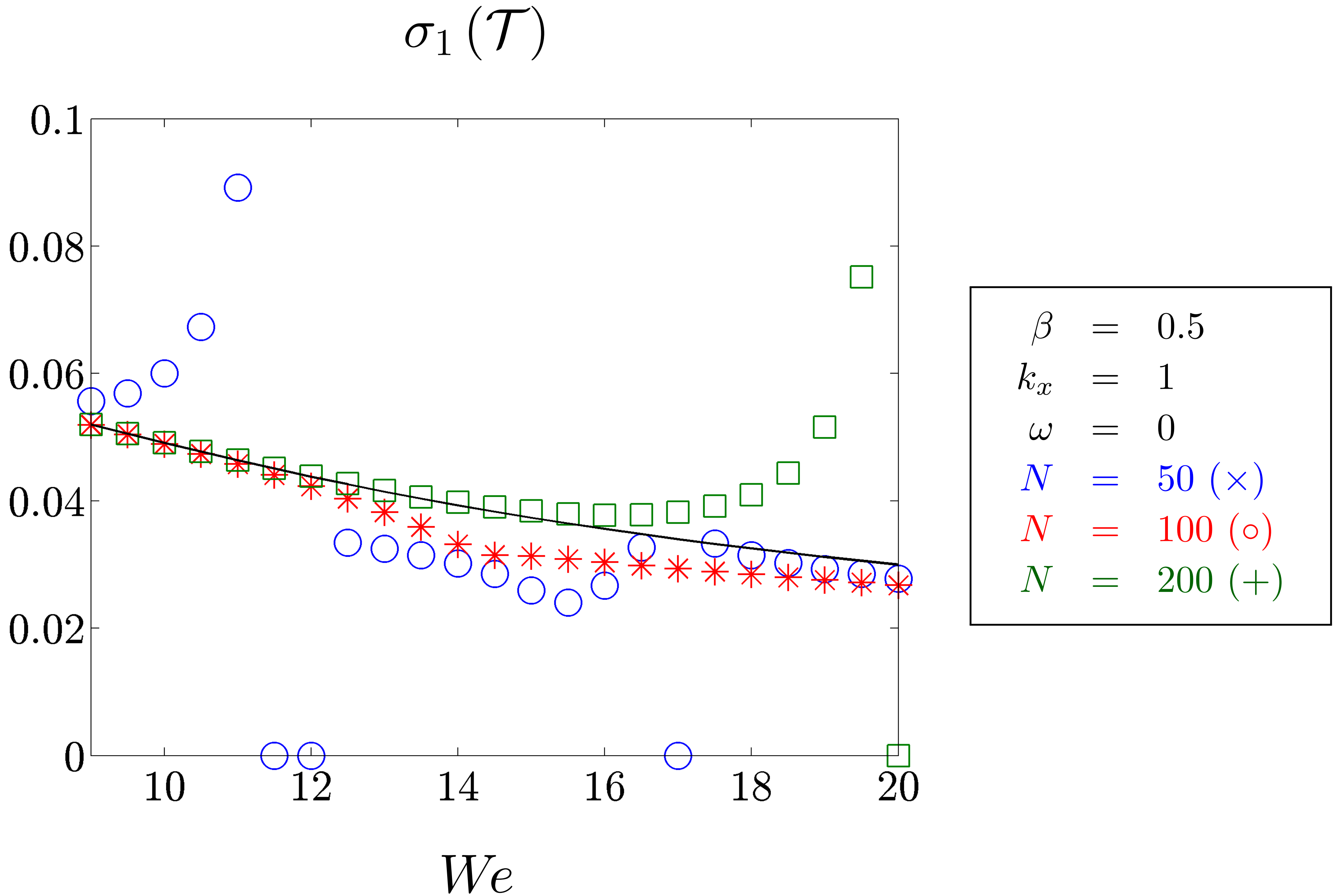
$$
\begin{aligned}
0 &= -\nabla p + (1-\beta)\,\nabla\cdot\boldsymbol{\tau} + \beta\,\nabla^2\mathbf{v} + \mathbf{d} \\
0 &= \nabla\cdot\mathbf{v} \\
\boldsymbol{\tau}_t &= \nabla\mathbf{v} + (\nabla\mathbf{v})^T - \boldsymbol{\tau} + {\color{blue}We}\,(\boldsymbol{\tau}\cdot\nabla\bar{\mathbf{v}} + \bar{\boldsymbol{\tau}}\cdot\nabla\mathbf{v} \\
&\quad + (\bar{\boldsymbol{\tau}}\cdot\nabla\mathbf{v})^T + (\boldsymbol{\tau}\cdot\nabla\bar{\mathbf{v}})^T - \mathbf{v}\cdot\nabla\bar{\boldsymbol{\tau}} - \bar{\mathbf{v}}\cdot\nabla\boldsymbol{\tau})
\end{aligned}
$$

$$
{\color{blue}We} = \frac{\text{polymer relaxation time}}{\text{characteristic flow time}}
$$

Largest singular value of $\mathcal{T}$



$\beta = 0.5$
$k_x = 1$
$\omega = 0$
$N = 50\ (\times)$
$N = 100\ (\circ)$
$N = 200\ (+)$


$\sigma_1(\mathcal{T})$
0.1
0.08
0.06
0.04
0.02
0
10
12
14
16
18
20
We
$\beta = 0.5$
$k_x = 1$
$\omega = 0$
$N = 50\ (\times)$
$N = 100\ (\circ)$
$N = 200\ (+)$

Input-output differential equations

- Frequency response operator

$$\mathcal{T}(\omega) : \left\{ \begin{array}{rcl} [\mathcal{A}_0\, \phi]\,(y) & = & [\mathcal{B}_0\, d]\,(y) \\ \varphi(y) & = & [\mathcal{C}_0\, \phi]\,(y) \\ 0 & = & \mathcal{N}_0\, \phi(y) \end{array} \right.$$

- Adjoint of the frequency response operator

$$\mathcal{T}^\star(\omega) : \left\{ \begin{array}{rcl} [\mathcal{A}_0^\star\, \psi]\,(y) & = & [\mathcal{C}_0^\star\, f]\,(y) \\ g(y) & = & [\mathcal{B}_0^\star\, \psi]\,(y) \\ 0 & = & \mathcal{N}_0^\star\, \psi(y) \end{array} \right.$$

Composition of $\mathcal{T}$ with $\mathcal{T}^\star$

- **Cascade connection**

$$
\mathcal{T}\mathcal{T}^\star : \begin{cases} [\mathcal{A}\,\xi]\,(y) \;=\; [\mathcal{B}\,f]\,(y) \\ \varphi(y) \;=\; [\mathcal{C}\,\xi]\,(y) \\ 0 \;=\; \mathcal{N}\,\xi(y) \end{cases}
$$

$\star$ **Do e-value decomposition of $\mathcal{T}\,\mathcal{T}^\star$ and $\mathcal{T}^\star\,\mathcal{T}$ in Chebfun**

$$
\left[\mathcal{T}(\omega)\,\mathcal{T}^\star(\omega)\,u_n(\cdot,\omega)\right](y) \;=\; \sigma_n^2(\omega)\,u_n(y,\omega)
$$

$$
\left[\mathcal{T}^\star(\omega)\,\mathcal{T}(\omega)\,v_n(\cdot,\omega)\right](y) \;=\; \sigma_n^2(\omega)\,v_n(y,\omega)
$$

Example: 1D heat equation

$$
\begin{aligned}
\phi_t(y,t) \;&=\; \phi_{yy}(y,t) \,+\, d(y,t), \qquad y \,\in\, [-1,\,1] \\
\phi(\pm 1,t) \;&=\; 0
\end{aligned}
$$

- Frequency response and adjoint operators

$$
\mathcal{T}(\omega) : \left\{ \begin{aligned}
\phi''(y) \,-\, \mathrm{j}\,\omega\,\phi(y) \;&=\; -d(y) \\
\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} E_{-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_{1} \right) \phi(y) \;&=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned} \right.
$$

$$
\mathcal{T}^{\star}(\omega) : \left\{ \begin{aligned}
\psi''(y) \,+\, \mathrm{j}\,\omega\,\psi(y) \;&=\; f(y) \\
g(y) \;&=\; -\psi(y) \\
\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} E_{-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_{1} \right) \psi(y) \;&=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned} \right.
$$

Integral form of a differential equation

Driscoll, J. Comput. Phys., 2010

- 1D diffusion equation: differential form

$$\left(D^{(2)} - \mathrm{j}\omega I\right)\phi(y) \;=\; -\,d(y)$$

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} E_{-1} \;+\; \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_{1}\right)\phi(y) \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Auxiliary variable: $\quad \nu(y) \;=\; \left[D^{(2)}\,\phi\right](y)$

Integrate twice

$$\phi'(y) \;=\; \int_{-1}^{y} \nu(\eta_1)\,\mathrm{d}\eta_1 \;+\; k_1 \;=\; \left[J^{(1)}\,\nu\right](y) \;+\; k_1$$

$$\begin{aligned} \phi(y) \;&=\; \int_{-1}^{y}\left(\int_{-1}^{\eta_2} \nu(\eta_1)\,\mathrm{d}\eta_1\right)\mathrm{d}\eta_2 \;+\; \begin{bmatrix} 1 & (y\,+\,1) \end{bmatrix}\begin{bmatrix} k_2 \\ k_1 \end{bmatrix} \\ &=\; \left[J^{(2)}\,\nu\right](y) \;+\; K^{(2)}\,\mathbf{k} \end{aligned}$$

- 1D diffusion equation: integral form

$$\left(I - \mathrm{j}\omega J^{(2)}\right)\nu(y) - \mathrm{j}\omega\, K^{(2)}\,\mathbf{k} = -\,d(y)$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} k_2 \\ k_1 \end{bmatrix} = -\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} E_{-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_1\right) J^{(2)}\nu(y)$$

Eliminate $\mathbf{k}$ from the equations to obtain

$$\left(I - \mathrm{j}\omega J^{(2)} + \frac{1}{2}\,\mathrm{j}\omega\,(y + 1)\,E_1 J^{(2)}\right)\nu(y) = -d(y)$$

☞ **More suitable for numerical computations than differential form**
integral operators and point evaluation functionals are well-conditioned

Online resources

- Chebfun

  http://www2.maths.ox.ac.uk/chebfun/

  Google: "chebfun"

- Computing frequency responses of PDEs

  http://www.umn.edu/~mihailo/software/chebfun-svd/

  Google: "frequency responses pde"

Summary

- method for computing frequency responses of PDEs
- easy-to-use mini-toolbox in MATLAB
  - ⋆ enabling tool: Chebfun
- two major advantages over currently available schemes
  - ⋆ avoids ill-conditioning of high-order differentiation matrices
  - ⋆ easy implementation of boundary conditions

Acknowledgments

TEAM:

Binh Lieu

SUPPORT:

NSF CAREER Award CMMI-06-44793

SOFTWARE:

http://www.umn.edu/∼mihailo/software/chebfun-svd/