On the Design of Optimal Structured and Sparse Feedback Gains via Sequential Convex Programming

Makan Fardad and Mihailo R. Jovanović

Abstract— We consider the problem of finding optimal feedback gains in the presence of structural constraints and/or sparsity-promoting penalty functions. Such problems are known to be difficult due to their lack of convexity. We provide an equivalent reformulation of the optimization problem such that its source of nonconvexity is isolated in one nonconvex matrix inequality of the form $Y \preceq X^{-1}$. Furthermore, we preserve the feedback gain as an optimization variable in the reformulated problem. Via linearizations of the nonconvex constraint, we introduce an iterative algorithm that solves a semidefinite program at every stage and for which the nonconvex constraint is satisfied upon convergence. We elaborate on the modular nature of the proposed scheme and show that it can be used in a wide range of network control problems.

Index Terms— Communication architecture, ℓ_1 minimization, optimization, semidefinite programming, sequential convex programming, sparsity-promoting control, structural constraints.

I. INTRODUCTION

The problem of designing optimal state and output feedback gains has been investigated since the 1960s [1], [2]. The linear quadratic regulator (LQR) [1] represents a case in which the optimal feedback gain can be found analytically. However, LQR has turned out to be the exception rather than the rule, in the sense that most optimal control problems do not permit a closed-form solution. Of particular interest are problems in which the feedback gain is constrained in some way, e.g., it is restricted to being an *output* feedback gain or adhering to a certain *structural pattern* on the location of its nonzero entries. Another problem of interest is what we refer to as *sparsity-promoting* optimal control, in which the objective function includes terms that penalize the number of nonzero entries of the matrix gain [3], [4]. The motivation for such problems comes from the desire to minimize the number of communication links between the many components of large-scale and networked control systems.

Recent work on the design of optimal controllers for classes of linear time invariant systems includes [5]–[15]. Particular attention has been paid to the problem of optimal structured control in [16]–[18], where the $\mathcal{H}_2$-norm of the closed-loop system is minimized among all controllers that respect a predetermined communication architecture. The problem of optimal sparse control has been considered in

[3], [4], [19]–[22], where a combination of $\mathcal{H}_2$-norm and sparsity-promoting penalty terms are minimized with the purpose of obtaining controllers with minimal internodal communication links.

The main contribution of this work is the development of a procedure for solving structured and/or sparse feedback gain design problems. Since these optimization problems are nonconvex in general, we propose an iterative procedure that solves a convex approximation of the original optimization problem at every stage. The distinctive features of our approach are the following:

- We provide an equivalent characterization of the optimization problem with the property that its source of nonconvexity is isolated in one nonconvex constraint of the form $Y \preceq X^{-1}$, where both of the positive definite matrices X and Y are optimization variables. This allows a variety of approximation/relaxation schemes to be brought to bear on the reformulated problem, in order to yield a convex program that can be solved efficiently. We explore one such approximation based on linearization and sequential convex programming.

- We avoid variable transformations that result in the loss of the feedback gain matrix as an optimization variable. By preserving the feedback gain as an optimization variable, we are able to directly enforce on it desired structural constraints and/or penalize its nonzero entries via sparsity-promoting cost functions.

In this paper we adopt a framework which resembles that in [4], [16]. We consider a standard linear quadratic optimal control problem in Section II and use it to develop an alternative computational algorithm. The main results of this work appear under Propositions 1 and Algorithm 1 in Section III. We apply these results to various optimal control problems of practical interest in Section IV. We offer an illustrative example in Section V.

II. PROBLEM FORMULATION

We consider linear time invariant systems described by

$$x_{k+1} = Ax_k + B_1 w_k + B_2 u_k \tag{1}$$
$$z_k = Cx_k + Du_k,$$

where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^n$ is the exogenous input, $u_k \in \mathbb{R}^m$ is the control input, and $z_k \in \mathbb{R}^{n+m}$ is the performance output, all evaluated at the discrete time instant k. The matrices C and D are such that z_k encapsulates information about both the state and the input at every k.

We assume $C = [\,Q^{1/2}\; 0\,]^T$ and $D = [\,0\; R^{1/2}\,]^T$, where $Q \succ 0$ and $R \succ 0$. We further assume that $B_1 B_1^T \succ 0$ and that (A, B_2) is stabilizable.

We seek feedback gains F,

$$u_k = -Fx_k,$$

that satisfy certain structural constraints and are optimal in the linear quadratic sense. With this choice of control, the closed-loop system is described by

$$x_{k+1} = (A - B_2 F)x_k + B_1 w_k \qquad (2)$$
$$z_k = \begin{bmatrix} Q^{1/2} \\ -R^{1/2}F \end{bmatrix} x_k.$$

Our work is centered around the problem of searching for feedback gains that minimize the $\mathcal{H}_2$ norm [23] of the closed-loop system (2). In particular, we search for a matrix F that solves

minimize $\quad \mathrm{trace}(B_1^T P B_1)$

subject to $\quad P - (A - B_2 F)^T P (A - B_2 F) = Q + F^T R F$

$\qquad\qquad P \succ 0.$
$$\text{(H2)}$$

Although we begin our developments by focusing on problem (H2) and its equivalent formulations, we do this in order to set the stage for the latter parts of the paper where (H2) is extended to problems which include structural constraints on F or problems in which the nonzero entries of F are penalized so as to render a sparse feedback gain. Indeed, while the optimal F that solves (H2) can be obtained from standard LQR theory, the procedure for solving the LQR problem does not lend itself to extensions which place additional restrictions or penalties on F.

III. MAIN RESULT: AN ITERATIVE ALGORITHM FOR SOLVING PROBLEM (H2)

In this section we reformulate (H2) such that its source of nonconvexity is isolated in one nonconvex matrix inequality constraint. This reformulation lends itself to approximations, while keeping the feedback gain F as an optimization variable. We then propose and justify an approximation scheme, and employ it in an iterative method introduced in Algorithm 1.

Proposition 1: The optimization problem (H2), with $Q \succ 0$ and $B_1 B_1^T \succ 0$, is equivalent to

minimize $\quad \mathrm{trace}(B_1^T X B_1)$

subject to $\quad \begin{bmatrix} X - Q - K & (A - B_2 F)^T \\ A - B_2 F & Y \end{bmatrix} \succeq 0$

$\qquad\qquad \begin{bmatrix} K & F^T \\ F & R^{-1} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$ $\qquad$ (J)

$\qquad\qquad Y \preceq X^{-1},$

where the optimization variables are the matrix F and the symmetric matrices K, X, Y.

Proof: See Appendix. ∎
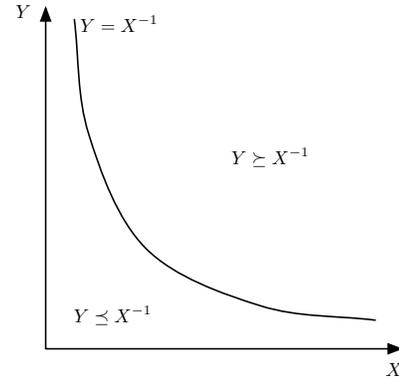
Fig. 1: The set described by the matrix inequality $[X, I; I, Y] \succeq 0$, or equivalently $Y \succeq X^{-1}$, is convex, whereas the one described by $Y \preceq X^{-1}$ is not.

We hereafter refer to (J) as the MI-equivalent (or matrix inequality equivalent) of problem (H2).

The only obstacle to finding the globally optimal solution of (J) is the nonconvex constraint $Y \preceq X^{-1}$; see Fig. 1. This is reminiscent of the results in [24], [25]. We work our way around the nonconvex constraint by using a method based on sequential convex programming [26]. We reformulate (J) by replacing the inequality constraint $Y - X^{-1} \preceq 0$ with the equality constraint $Y - X^{-1} = Z_+ - Z_-$, where Z_+ and Z_- are both positive semidefinite matrices. Moreover, we penalize the positive component Z_+ by adding the term $\lambda\,\mathrm{trace}(Z_+)$, $\lambda > 0$ to the objective. We thus have

minimize $\quad \mathrm{trace}(B_1^T X B_1) + \lambda\,\mathrm{trace}(Z_+)$

subject to $\quad \begin{bmatrix} X - Q - K & (A - B_2 F)^T \\ A - B_2 F & Y \end{bmatrix} \succeq 0$

$\qquad\qquad \begin{bmatrix} K & F^T \\ F & R^{-1} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$

$\qquad\qquad Y - X^{-1} = Z_+ - Z_-, \quad Z_+ \succeq 0, \quad Z_- \succeq 0,$

where the optimization variables are the matrices F, K, X, Y, Z_+, Z_-. Clearly, as λ grows, the matrix Z_+ approaches zero and the original matrix inequality $Y - X^{-1} \preceq 0$ is recovered. Thus for large enough λ, the minimizer of this problem is equal to that of (J), [26, pp. 14].

Finally, we linearize the (still nonconvex) equality constraint $Y - X^{-1} = Z_+ - Z_-$ around our current best estimate $\mathcal{X}$ of X to obtain

$$Y - h_{\mathcal{X}}(X) = Z_+ - Z_-,$$

where

$$h_{\mathcal{X}}(X) := \mathcal{X}^{-1} - \mathcal{X}^{-1}(X - \mathcal{X})\mathcal{X}^{-1}.$$
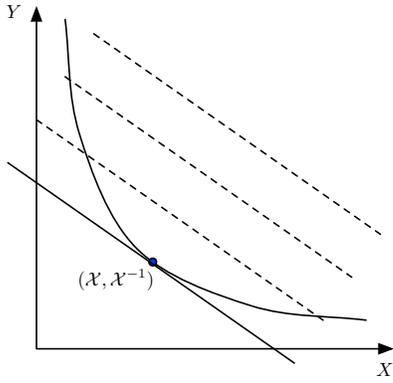
Fig. 2: The solid line depicts the affine approximation $Y = h_{\mathcal{X}}(X)$ of $Y = X^{-1}$ around the point $(\mathcal{X}, \mathcal{X}^{-1})$. The dashed lines depict the contour lines corresponding to $\mathrm{trace}(Z_+)$; the value of $\mathrm{trace}(Z_+)$ increases as one moves from bottom-left to top-right of the picture.

We now iteratively solve the problem

$$\text{minimize} \quad \mathrm{trace}(B_1^T X B_1) + \lambda\,\mathrm{trace}(Z_+)$$

$$\text{subject to} \quad \begin{bmatrix} X - Q - K & (A - B_2 F)^T \\ A - B_2 F & Y \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} K & F^T \\ F & R^{-1} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$$

$$Y - \mathcal{X}^{-1} + \mathcal{X}^{-1}(X - \mathcal{X})\mathcal{X}^{-1} = Z_+ - Z_-$$

$$Z_+ \succeq 0, \quad Z_- \succeq 0,$$

(RLX)

where at every iteration we take $\mathcal{X}$ to be the solution of (RLX) at the previous step. Furthermore, we increase the value of λ at every iteration by setting $\lambda := \mu\,\lambda$ for some fixed $\mu > 1$. The procedure is initialized with $\mathcal{X} := I$ and some small value of $\lambda > 0$. We summarize this in Algorithm 1.

Algorithm 1 Iterative algorithm for solving (J)

1: **given** $\lambda_0 > 0$, $\mu > 1$, and $\varepsilon > 0$.
2: **for** $i = 1, 2, \ldots$ **do**
3:      If $i = 1$, set $\lambda := \lambda_0$, set $\mathcal{X} := I$.
4:      If $i > 1$, set $\lambda := \mu\,\lambda$, set $\mathcal{X}$ equal to optimal X from previous iteration.
5:      Solve (RLX) to obtain X^*, Y^*.
6:      If $\|X^* - \mathcal{X}\| < \varepsilon$, $\|Y^* - (X^*)^{-1}\| < \varepsilon$, **quit**.
7: **end for**

We next elaborate on the operation of Algorithm 1. Let X_J denote the optimal value of X resulting from the solution of (J). Then, for small enough values of λ the feasible set of (RLX) contains that of (J), and in particular contains X_J. (This is due fact that when λ is very small, there is no penalty on the size of Z_+, and the equality constraint in (RLX) can be eliminated without affecting the solution of the optimization problem.) Hence, during the initial iterations of Algorithm 1, when λ has small values,

the optimal value of (RLX) renders a lower bound on the optimal value of (J). As the iterations progress and the value of λ grows, the matrix Z_+ approaches zero and the equality constraint in (RLX) implies $Y - h_{\mathcal{X}}(X) = -Z_- \preceq 0$. This, together with $Y - X^{-1} \succeq 0$, forces $Y = X^{-1}$; see Fig. 2.

We make no claim on the convergence of Algorithm 1 or the global optimality of the solution that results from it. However, in our extensive numerical experiments this algorithm always converges and its solution is identical to that of (H2). We emphasize that there is no requirement on the open-loop stability of the system in order for our results to hold.

At this point, one may wonder what is the advantage of reformulating (H2) as (J), in particular since the solution of (H2) – the standard LQR gain, obtained by solving an algebraic Riccati equation – has been known for decades. To justify the formulation (J) and its relaxation (RLX), we note that even though the nonconvex problem (H2) can be solved using LQR theory, it is not at all obvious how to obtain optimal solutions once either the objective function or the constraints of (H2) are modified.

As an example of such a modification, consider the problem of obtaining a *sparse* feedback gain F by including additional terms in the objective that penalize the nonzero entries of F so as to promote its sparsity [3], [4], [20], [27]. As another example, one may be interested in obtaining a *structured* feedback gain F by including additional constraints that enforce a desired architecture on the zero entries of F, [16], [18]. In the new framework proposed here, any convex penalty functions or constraints can be incorporated into (RLX) without affecting the iterative optimization algorithm. In Section IV we give concrete instances of such scenarios and formulate their corresponding MI-equivalent and convex relaxations, as needed for the implementation of Algorithm 1.

IV. DESIGN OF OPTIMAL STRUCTURED/SPARSE FEEDBACK GAINS

A. Optimal Structured Feedback Gains

The problem of designing structured feedback gains was considered in [16], [18], where alternating methods and descent algorithms were used to obtain locally optimal solutions. In this section we introduce an iterative procedure, based on the MI-equivalent, for finding optimal structured feedback gains.

Consider the problem

$$\text{minimize} \quad \mathrm{trace}(B_1^T P B_1)$$

$$\text{subject to} \quad P - (A - B_2 F)^T P (A - B_2 F) = Q + F^T R F$$

$$F_{ij} = 0 \quad \text{if} \quad (i, j) \in \mathcal{S},$$

(3)

where the set $\mathcal{S}$ encapsulates the structural constraints imposed on the matrix F. More precisely, $\mathcal{S}$ is composed of index pairs such that if $(i, j) \in \mathcal{S}$ then subsystem i is *not* allowed to receive information from subsystem j. This is equivalent to the constraint $F_{ij} = 0$. A notationally compact

way to represent all such constraints on the entries of the matrix F is
$$E_{\mathcal{S}} \circ F = 0,$$
where $\circ$ denotes elementwise matrix multiplication and $E_{\mathcal{S}}$ is a matrix of the same dimension as F and defined as
$$(E_{\mathcal{S}})_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{S}, \\ 0 & \text{if } (i,j) \notin \mathcal{S}. \end{cases}$$
The above optimization problem is therefore equivalent to

minimize $\text{trace}(B_1^T P B_1)$

subject to $P - (A - B_2 F)^T P (A - B_2 F) = Q + F^T R F$

$\qquad\qquad E_{\mathcal{S}} \circ F = 0.$
$$\tag{4}$$

The structural constraints $E_{\mathcal{S}} \circ F = 0$ are affine and therefore convex. Thus we can apply to (4) the same relaxation procedure that we applied to (H2) and (J) to obtain (RLX). Doing so, we obtain

minimize $\text{trace}(B_1^T X B_1) + \lambda \, \text{trace}(Z_+)$

subject to $\begin{bmatrix} X - Q - K & (A - B_2 F)^T \\ A - B_2 F & Y \end{bmatrix} \succeq 0$

$\qquad\qquad \begin{bmatrix} K & F^T \\ F & R^{-1} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$

$\qquad\qquad Y - \mathcal{X}^{-1} + \mathcal{X}^{-1}(X - \mathcal{X})\mathcal{X}^{-1} = Z_+ - Z_-$

$\qquad\qquad Z_+ \succeq 0, \quad Z_- \succeq 0, \quad E_{\mathcal{S}} \circ F = 0.$
$$\tag{5}$$

We emphasize that, assuming there exist stabilizing F that satisfy the structural constraints, every step of the proof of Proposition 1 can be extended to show that problem (4) can be replaced with its MI-equivalent. The latter problem is then approximated by (5), obtained by replacing the nonconvex constraint $Y \preceq X^{-1}$ with the convex constraint $Y - h_{\mathcal{X}}(X) = Z_+ - Z_-$ and penalizing $\text{trace}(Z_+)$ in the objective. Furthermore, the iterative procedure described in Section III can be applied to the optimal structured problem by solving (5) in Step 5 of Algorithm 1 [rather than solving (RLX)].

B. Optimal Sparse Feedback Gains

The problem of designing sparse feedback gains was considered in [3], [4], [19], where the alternating direction method of multipliers and reweighted ℓ_1 relaxations were used to obtain locally optimal solutions. In this section we introduce an iterative procedure, based on the MI-equivalent, for finding optimal sparse feedback gains.

Consider the problem

minimize $\text{trace}(B_1^T P B_1) + \gamma \|W \circ F\|_{\ell_1}$

subject to $P - (A - B_2 F)^T P (A - B_2 F) = Q + F^T R F,$
$$\tag{6}$$

where W is a known weighting matrix of the same dimension as F, $\circ$ denotes elementwise matrix multiplication, and the weighted ℓ_1-norm in the objective function is intended to promote the sparsity of the matrix F. Let M be a matrix of the same dimension as F and with entries all equal to one. Clearly, if $W = M$ then we recover the standard ℓ_1-norm, $\|M \circ F\|_{\ell_1} = \|F\|_{\ell_1} = \sum_{i,j} |F_{ij}|$. The sparsity-promoting properties of the (reweighted) ℓ_1-norm have been demonstrated previously in [28], [29], mostly in the context of solving underdetermined systems of linear equations with sparse solutions.

Using standard methods used in ℓ_1 optimization problems, it can be shown [3], [30] that the above optimization problem is equivalent to

minimize $\text{trace}(B_1^T P B_1) + \gamma \, \text{trace}(M^T V)$

subject to $P - (A - B_2 F)^T P (A - B_2 F) = Q + F^T R F$

$\qquad\qquad -V \leq W \circ F \leq V,$
$$\tag{7}$$

where the inequality constraints $-V \leq W \circ F$ and $W \circ F \leq V$ are elementwise. These inequality constraints are affine and therefore convex, and $\text{trace}(M^T V)$ is also convex. Thus we can apply to (7) the same relaxation procedure that we applied to (H2) and (J) to obtain (RLX). Doing so, we obtain

minimize $\text{trace}(B_1^T X B_1) + \gamma \, \text{trace}(M^T V) + \lambda \, \text{trace}(Z_+)$

subject to $\begin{bmatrix} X - Q - K & (A - B_2 F)^T \\ A - B_2 F & Y \end{bmatrix} \succeq 0$

$\qquad\qquad \begin{bmatrix} K & F^T \\ F & R^{-1} \end{bmatrix} \succeq 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0$

$\qquad\qquad Y - \mathcal{X}^{-1} + \mathcal{X}^{-1}(X - \mathcal{X})\mathcal{X}^{-1} = Z_+ - Z_-$

$\qquad\qquad Z_+ \succeq 0, \quad Z_- \succeq 0, \quad -V \leq W \circ F, \quad W \circ F \leq V.$
$$\tag{8}$$

Once again, every step of the proof of Proposition 1 can be extended to show that problem (7) can be replaced with its MI-equivalent. The latter problem is then approximated with (8), obtained by replacing the nonconvex constraint $Y \preceq X^{-1}$ with the convex constraint $Y - h_{\mathcal{X}}(X) = Z_+ - Z_-$ and penalizing $\text{trace}(Z_+)$ in the objective. And the iterative procedure described in Section III can be applied to the optimal structured problem by solving (8) in Step 5 of Algorithm 1 [rather than solving (RLX)].

In our numerical experiments of solving (8) as part of Algorithm 1, even the simple choice $W = M$ renders sparse feedback gains F. However, it is possible to use a *reweighted* ℓ_1 algorithm [3], [29] that iteratively adapts the matrix W so that the least significant entries of F are forced to become zero and the nonzero entries of F are readjusted accordingly. We summarize this scheme in Algorithm 2.

In closing, we point out that problem (6) and Algorithm 2 can be used to 'identify' the most effective sparsity pattern in F. Then, once this pattern has been identified, one can build it into the matrix $E_{\mathcal{S}}$ of Section IV-A and solve (4) to obtain the optimal structured feedback matrix F. The latter step is called 'polishing', as it further refines the optimal sparse F found from (6). This two-step procedure was introduced in [4], [31].

V. ILLUSTRATIVE EXAMPLE

In this section we present a simple example. All computations were performed using CVX, a package for specifying

Algorithm 2 Reweighted ℓ_1 algorithm

1: **given** $\gamma > 0$, $\delta > 0$ and $\epsilon > 0$.
2: **for** $r = 1, 2, \ldots$ **do**
3:     If $r = 1$, set $\mathcal{F} := 0$, set $W_{ij} := 1$, form matrix W.
4:     If $r > 1$, set $\mathcal{F}$ equal to optimal F from previous iteration, set $W_{ij} := 1/(|\mathcal{F}_{ij}| + \delta)$, form matrix W.
5:     Invoke Algorithm 1, replacing (RLX) in Step 5 with (8), to obtain F^*.
6:     If $\|F^* - \mathcal{F}\| < \epsilon$, **quit**.
7: **end for**

and solving convex programs [32], [33].

We consider problem (1) in which the state-space parameters are randomly selected as

$$A = \begin{bmatrix} 0.9504 & 0.0134 & -0.0176 & -0.0169 & 0.0231 \\ -0.0341 & 1.0817 & -0.0290 & 0.0306 & -0.0446 \\ 0.0312 & -0.0576 & 1.0635 & -0.0810 & 0.0065 \\ -0.0570 & 0.0313 & 0.0455 & 0.9892 & -0.0263 \\ -0.0368 & -0.0378 & -0.1045 & -0.0166 & 0.9321 \end{bmatrix},$$

$$Q = \begin{bmatrix} 6.2075 & 2.2502 & -2.9028 & -0.6915 & -1.9053 \\ 2.2502 & 7.9155 & -0.2435 & 2.9279 & -4.6740 \\ -2.9028 & -0.2435 & 2.2141 & 0.4269 & 0.2727 \\ -0.6915 & 2.9279 & 0.4269 & 2.8999 & -2.4567 \\ -1.9053 & -4.6740 & 0.2727 & -2.4567 & 3.6376 \end{bmatrix},$$

$$B_1 = \mathrm{diag}([\,-0.3843 \quad -0.4579 \quad -0.2915 \quad -0.3012 \quad -1.5886\,]),$$

$$B_2 = \mathrm{diag}([\,+1.0943 \quad +1.3242 \quad -0.1265 \quad -0.7372 \quad +0.2137\,]),$$

and $R = I$. We emphasize that there is no reason for B_1, B_2, or $R \succ 0$ to be diagonal, or for B_2 to be square; the proposed algorithms work just as well if these parameters are randomly chosen matrices. We have

$$\mathrm{eig}(A) = \begin{cases} 0.9185 + 0.0278i \\ 0.9185 - 0.0278i \\ 1.0606 + 0.0534i \\ 1.0606 - 0.0534i \\ 1.0589 \end{cases}$$

and thus the discrete-time system is open-loop unstable. For all computations we set $\lambda_0 = 1$, $\mu = 1.01$, and $\varepsilon = 5 \times 10^{-4}$.

- *Standard $\mathcal{H}_2$ problem:* The solution of (H2) using Algorithm 1, is

$$F^* = \begin{bmatrix} 0.7567 & 0.0631 & -0.3485 & -0.0749 & -0.1554 \\ 0.0114 & 0.7546 & 0.0419 & 0.1396 & -0.2719 \\ -0.0246 & 0.0684 & -1.5510 & 0.1676 & 0.3148 \\ 0.1029 & -0.1299 & 0.2910 & -0.8162 & 0.3948 \\ -0.0585 & -0.0290 & -0.6678 & -0.0722 & 0.3456 \end{bmatrix}$$

which is *identical* to the solution of the standard LQR problem. We have $\mathrm{trace}(B_1^T X^* B_1) = 17.50$.

- *Structured $\mathcal{H}_2$ problem:* The solution of (3) using Algorithm 1, in which (5) is solved in Step 5, is

$$F^* = \begin{bmatrix} 0.7569 & -0.1145 & -0.3213 & -0.2611 & 0 \\ 0.0155 & 0.8047 & 0.0333 & 0.1912 & -0.3164 \\ -0.0286 & 0.1248 & -1.5600 & 0.2291 & 0.2653 \\ 0.0990 & -0.0387 & 0.2747 & -0.7290 & 0.3213 \\ 0 & -0.0777 & -0.6726 & -0.1624 & 0.3946 \end{bmatrix}$$

where the structure imposed on F is

$$F_{1,5} = F_{5,1} = 0.$$

We have $\mathrm{trace}(B_1^T X^* B_1) = 18.07$.

- *Sparse $\mathcal{H}_2$ problem:* The solution of (6) using Algorithm 1 in which (8) is solved in Step 5, is

$$F^* = \begin{bmatrix} 0.7268 & 0.0000 & -0.2704 & -0.0348 & -0.1534 \\ 0.0000 & 0.7992 & 0.0000 & 0.1320 & -0.2730 \\ 0.0000 & 0.0000 & -1.5832 & 0.0707 & 0.3484 \\ 0.0000 & -0.0000 & 0.1331 & -0.7719 & 0.3816 \\ -0.0000 & -0.0000 & -0.4460 & -0.0416 & 0.2972 \end{bmatrix}$$

where the parameters γ and W are chosen to be

$$\gamma = 1/4, \quad W = \mathbb{1}\mathbb{1}^T - I.$$

We have $\mathrm{trace}(B_1^T X^* B_1) = 17.61$.

VI. CONCLUSION

We propose an iterative scheme for solving optimal control problems with structural constraints and/or sparsity-promoting penalty functions. The scheme solves a sequence of approximate convex optimization problems to arrive at a suboptimal solution to the original (nonconvex) problem. We demonstrate the effectiveness of our procedure using an illustrative example.

In our future work we will investigate the extension of the methods proposed here to the design of optimal output feedback controllers subject to structural constraints and/or sparsity-promoting penalty functions (as opposed to the state feedback controllers considered in this work).

VII. APPENDIX: PROOF OF PROPOSITION 1

The proof can be separated it into proving each of the following statements.

(a) Problem (H2) is equivalent to

minimize $\quad \mathrm{trace}(B_1^T X B_1)$
subject to $\quad X - (A - B_2 F)^T X (A - B_2 F) \succeq Q + F^T R F$
$\qquad\qquad X \succ 0.$

(9)

(b) Problem (9) is equivalent to

minimize $\quad \mathrm{trace}(B_1^T X B_1)$
subject to $\quad X - (A - B_2 F)^T X (A - B_2 F) \succeq Q + K$
$\qquad\qquad K \succeq F^T R F, \quad X \succ 0.$

(10)

(c) Problem (10) is equivalent to

minimize $\quad \mathrm{trace}(B_1^T X B_1)$
subject to $\quad X - (A - B_2 F)^T Y^{-1} (A - B_2 F) \succeq Q + K$
$\qquad\qquad K \succeq F^T R F, \quad X \succeq Y^{-1}, \quad Y \preceq X^{-1}$
$\qquad\qquad X \succ 0, \quad Y \succ 0,$

(11)

which follows from introducing $Y = X^{-1}$ and replacing the equality constraint $X = Y^{-1}$ by the pair of inequality constraints $X \succeq Y^{-1}$, $X \preceq Y^{-1}$.

(d) Problem (11) is equivalent to problem (J), which follows from multiple applications of the Schur complement.

The details of each step are omitted due to space limitations and will be reported elsewhere.

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