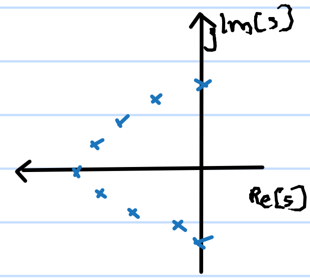
EE 8215 : Nonlinear Systems
Lecture 8 : Feb 16^{th} 2016, Tue

Last time: – Center Manifold Theory

Translation : Tool for stability of $\dot{x} = f(x)$ where linearisation is marginally stable

i.e. $A = \frac{\partial f}{\partial x}\Big|_{\bar{x}}$ has k eigenvalues on the $j\omega$ axis, $n-k$ eigenvalues in the LHP



$$\left.\begin{aligned} \dot{y} &= A_1 y + g_1(y,z) \\ \dot{z} &= A_2 z + g_2(y,z) \end{aligned}\right\} \circledast$$

Stability of $\circledast$ determined by stability of : $\dot{y} = A_1 y + g_1(y, h(y))$

$z = h(y)$: center manifold

Today: Ch. 3 $\longrightarrow$ Mathematical background

- Nonlinears ODEs : • RHS has to satisfy certain properties for a solution to exist
  - If soln exists, is it unique?

- Linear Algebra : $A\vec{x} = \vec{b}$
  - If $\vec{b}$ is in the range space of A, then $A\vec{x} = \vec{b}$ has a solution i.e. if $\vec{b} \in$ span {columns of A}

$\rightarrow$ Existence & uniqueness of solutions of $\dot{x} = f(x,t)$ time-varying | $\dot{x} = A(t)x$

or $\dot{x} = f(x)$ time-invariant | $\dot{x} = Ax$

- For $\dot{x} = Ax \Rightarrow x(t) = x(0)e^{At} \longrightarrow$ always has a solution
- For $\dot{x} = A(t)x \longrightarrow$ has a solution if $A(t)$ is a piecewise continuous function of time
  - $\rightarrow$ every element of $A(t)$ is piecewise continuous
  - $\rightarrow$ can have jumps, but not infinitely many of them

- We will consider functions f that have piecewise continuous dependence on t

Q1) What about dependence on x?

Q2) Is continuity enough?

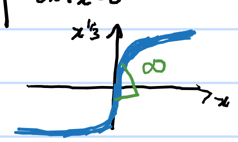
Eg ①: $\dot{x} = f(x) = x^{1/3} \longrightarrow$ no explicit time dependence; continuous.

- For $x(0) = 0 \Rightarrow x(t) = 0$ is a solution
- Can show that : $x(t) = \left(\frac{2t}{3}\right)^{3/2}$ is also a solution

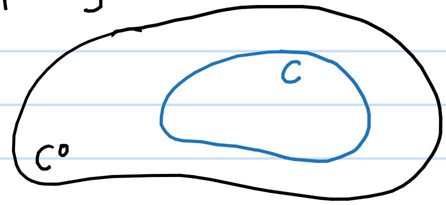
$\therefore$ Need to restrict a class of functions f a bit more ...

Source of trouble: Infinite slope of $f(x)$ @ the origin

$\frac{\partial f}{\partial x}\Big|_{x=0} = \infty$



- Graphically:



C^0 : continuous functions.

↳ How much do we need to restrict C^0 to get a unique solution?

➤ Fact ① If f is continuous $\Rightarrow$ existence of solutions on $[0, t_f)$

$\cancel{\Rightarrow}$ uniqueness

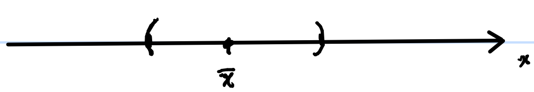
↳ For uniqueness, need Lipschitz continuity

$$\|f(x,t) - f(y,t)\| \leq L\|x-y\| \qquad (L)$$

- If (L) holds $\forall t$ and $\forall$ points in a certain neighbourhood of $\bar{x} \in \mathbb{R}^n$ for some constant L, the $f(x)$ is locally Lipschitz continuous. (in x)
- If (L) holds $\forall\ x, y \in \mathbb{R}^n \Rightarrow$ globally Lipschitz continuous.

➤ Eg: ②. $f(x) = x^2$

$$|f(x) - f(y)| = |x^2 - y^2| = |(x+y)(x-y)| = |x+y|\,|x-y| \leq L\,|x-y|$$



$\therefore$ Locally Lipschitz continuous.

But not globally $\because$ larger x, y requires larger L

➤ Eg ③: $f(x) = x^3$

$$|x^3 - y^3| = |x^2 + xy + y^2|\,|x-y|$$

Locally Lipschitz continuous but not globally

Alternative way of checking:

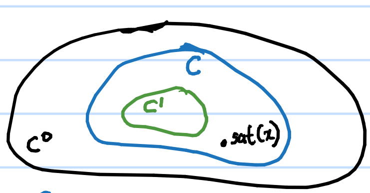
Check if $f(x)$ is continuously differentiable.

↳ Back to eg ②: $f(x) = x^2 \Rightarrow \frac{\partial f}{\partial x} = 2x$

$f(x) = x^3 \Rightarrow \frac{\partial f}{\partial x} = 3x^2$ } Both continuously differentiable.

↳ $f(x)$ continuous; $f(x)$ differentiable + $\frac{df}{dx}$ continuous

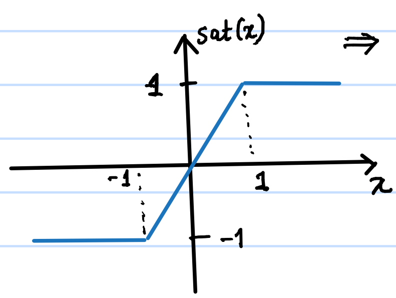
$\left|\frac{\partial f}{\partial x}\right| \leq L$ locally | in both cases, such L exists locally (but not globally).



C : Lipschitz continuous

C' : continuously differentiable

➤ Eg ④:



$\Rightarrow$ Globally Lipschitz continuous but not continuously differentiable

$\rightarrow \because$ has kinks

Fact ②: If f is locally Lipschitz continuous $\Rightarrow$ existence and uniqueness on some finite time interval $[0, t_f)$

③ If f is globally Lipschitz continuous $\Rightarrow$ existence & uniqueness on $[0, \infty)$ (strong req'ment)

Eg ⑤: $f(x) = -x^3$

- Not globally Lipschitz continuous
- But globally continuous

} sufficient condition but not a necessary condition

Eg ⑥: Linear System: $\dot{x} = A(t)\,x$

$$\|f(x,t) - f(y,t)\| = \|A(t)(x-y)\|$$
$$\leq \underbrace{\|A(t)\|_{ind}}_{\text{induced norm}} \|x-y\|$$

If A is constant $\rightarrow$ always have a bounded induced norm

If A is piecewise continuous $\rightarrow$ compute supremum over a time interval

Q3) How about continuity w.r.t. ICs? (continuous dependence on ICs)

↳ If two systems start close enough, do they stay close enough?

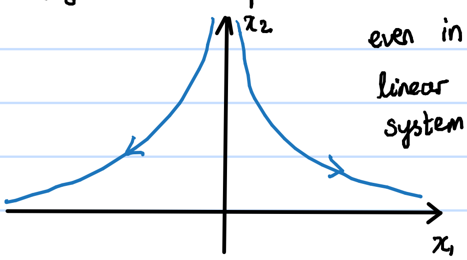
Fact ④: If existence & uniqueness $\Rightarrow$ continuous dependence on ICs on finite time intervals

- Given f: locally Lipschitz continuous:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t. } \|x_0 - y_0\| < \delta \Rightarrow \|\phi(x_0, t) - \phi(y_0, t)\| < \varepsilon \quad \forall t \in [0, t_f)$$

Global requirement ($\forall t$) too much to hope for:

Eg: saddle point



even in linear systems.

Eg: chaos

no continuous dependence on ICs $\forall t$

Q4) How about continuous dependence w.r.t. parameters?

$\dot{x} = f(x, t, \mu)$ $\qquad$ μ: const. parameter; can be vector-valued

⇕ equivalent

$\dot{x} = f(x, \mu, t)$ $\quad \Rightarrow \quad$ If $z = \begin{bmatrix} x \\ \mu \end{bmatrix}$

$\dot{\mu} = 0$ $\qquad\qquad\qquad\qquad$ $\dot{z} = g(z, t)$

Sensitivity of solutions w.r.t parameters:

Back to fact 4; ?

- Given $\dot{x} = f(x, \mu, t)$
- Assume - f is a continuous function of μ
  - f is continuously differentiable in a neighbourhood of some constant value $\bar{\mu}$
- Given $\bar{\mu}$ for which this holds → can conclude existence & uniqueness on $[t_0, t_f)$

$\therefore$ continuous dependence on parameters.