Lecture 20

April 17 /2014

Stability of time-varying systems

$\dot{x} = f(x,t)$
↑ time appears explicitly in the state equation.

Linear systems: $\boxed{\dot{x} = A(t)x}$

Class $\underbrace{K, K_\infty}_{\alpha(r)}$; $\underbrace{K_L}_{\beta(r,s)}$ functions : (are scalar valued)

↓

System theoratic interpretation of our symbols :

$r: \|x_0\|$, $s: t - t_0$

A Continuous function $\alpha : [0,\infty) \longrightarrow [0,\infty)$ is a class K
(function: scalar argument & scalar valued!)

function if :
1°) $\alpha(0) = 0$
2°) It's strictly increasing

If in addition to 1° & 2° : $\alpha(r) \xrightarrow{r\to\infty} +\infty$ then α is a K_∞ function.

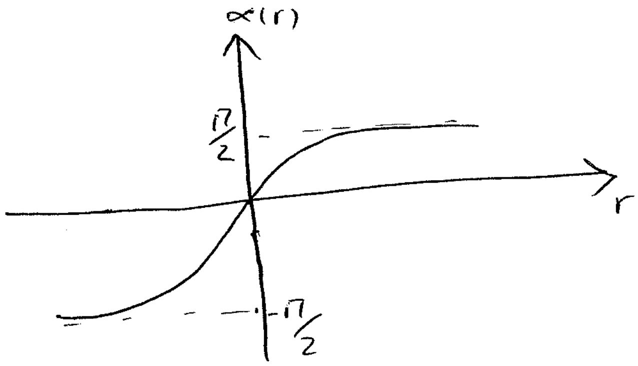
Ex 1 : $\alpha(r) = r^c$; $c>0$

$\boxed{\alpha(0) = 0}$

$\frac{d\alpha}{dr} = cr^{c-1} > 0 \rightarrow$ strictly increasing

$\lim_{r\to\infty} r^c = \infty \longrightarrow \boxed{K_\infty}$

Ex2: $\alpha(r) = a\tan r$



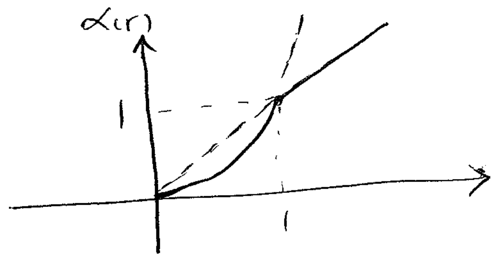
$\rightarrow$ $\alpha(0) = 0$

$\frac{d\alpha}{dr} > 0 = \frac{1}{1+r^2}$ $\Big\}$ $\rightarrow$ Class K

but not K_∞!

Ex3

$\alpha(r) = \min(r, r^2)$



$\rightarrow$ K_∞

So, K & K_∞ functions don't need to be differentiable!

$\beta(r,s) : [0,\infty) \times [0,\infty) \rightarrow [0,\infty)$

1°) $\beta(\cdot, s)$: Class K function

(For fixed 's', $\beta(\cdot, s)$ is class K)

2°) $\beta(r, \cdot)$ is decreasing and $\beta(r,s) \xrightarrow{s\to\infty} 0$ for any fixed r.

Ex4 $\beta(r,s) = kr^c e^{-as}$, $c, k, a > 0$

$\longrightarrow$ $\boxed{\text{KL function}}$

Ex 5 $\quad \beta(r,s) = \dfrac{r}{krs+1}$, $k>0$

$\beta(0,s) = 0 \rightarrow$ first condition.

Checking the second condition: Is it increasing?

$$\frac{d\beta}{dr} = \frac{krs+1-krs}{(krs+1)^2} > 0 \rightarrow \boxed{\text{Class } K}$$

"strictly increasing" (for fixed s)
(monotonically)

$\rightarrow \beta(r,\cdot)$, for fixed r, is decreasing and goes to 0 as $s \rightarrow \infty$

$\Rightarrow \boxed{K_L \text{ Function}}$

$\bar{x}=0$ is a stable eq. point of $\dot{x} = f(x,t)$ iff $\forall \varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0)$ s.t. $\|x_{(t_0)}\| < \delta \rightarrow \|x(t)\| < \varepsilon$

$\varphi(x(t_0), t_0)$

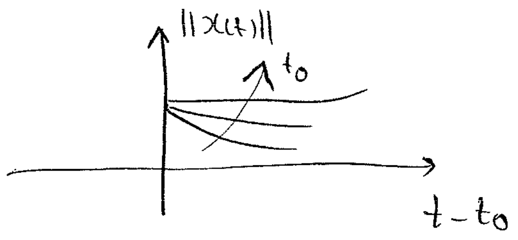
If $\delta = \delta(\varepsilon)$ (doesn't depend on t_0) then uniform stability!

EX: $\quad \dot{x} = -\underbrace{\dfrac{1}{1+t}}_{a(t)} x \Rightarrow x(t) = \dfrac{1+t_0}{1+t} x(t_0)$, $t \geq t_0$

$\Rightarrow$ origin is uniformly stable because $\rightarrow \|x(t)\| \leq \|x(t_0)\|$ for all $t \geq t_0$

Also, $\lim_{t \to \infty} \|x(t)\| = 0 \rightarrow$ asymptotically stable!

Is it uniformly asymptotically Stable?



↳ Is the rate of convergance going to be different when we change t_0?

$$\Rightarrow \quad x(t) = \frac{1}{1 + \frac{t - t_0}{1 + t_0}} \, x(t_0) \quad \rightarrow \quad t_0 \uparrow, \text{ rate of convergance } \uparrow$$

→ not uniformly asymptotically stable!

(increase with t_0)

So, ⇒ no uniform convergance rate

Stability Characterization in terms of K & K_L functions:

The origin of $\dot{x} = f(x, t)$ is

1°) uniformly Stable if there is a Class K function (C is not function of t_0)

$\alpha(\cdot)$ and a Constant $C > 0$

s.t. $\|x(t)\| \leq \alpha(\|x_0\|)$ for all $t \geq t_0$ and all $x(t_0)$ with $\|x(t_0)\| < c$

2°) uniformly asymptotically Stable if there is a class K_L function $B(\cdot, \cdot)$ and a constant $\underline{c}$ s.t.

$\|x(t)\| \leq B(\|x(t_0)\|, t - t_0)$ for all $t \geq t_0$ and all

$x(t_0)$ with $\|x(t_0)\| < c$

Lyapunov functions for time-varying systems:

<u>Ex</u>: $\dot{x} = A(t)\,x$

$\Rightarrow V(x) = x^T P(t)\, x$

<u>Ex</u> $\dot{x} = f(x,t)$

$V(x,t) \Rightarrow \dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \underbrace{f(x,t)}_{\dot{x}}$

<u>Comparison Functions</u>

what's coming on Tuesday?

A big picture overview of

$\alpha_1(\|x\|) \leq V(x,t) \leq \alpha_2(\|x\|)$

where α_1, α_2 are class K (or K_∞) functions!

<u>Aside Ex</u>: $V(x) = x^T P x$; $P \neq P(t)$

$\lambda_{min(P)} \|x\|^2 \leq x^T P x \leq \lambda_{max(P)} \|x\|^2$

$\rightarrow \alpha_1(r) = \lambda_{min}(P) r^2$, $\alpha_2(r) = \lambda_{max}(P) r^2 \rightarrow$

which are K_∞ if P, is PD!