Lecture 15

March 25

Before the spring break:

- Lyapunov Direct Method

$V(x)$: positive Definite

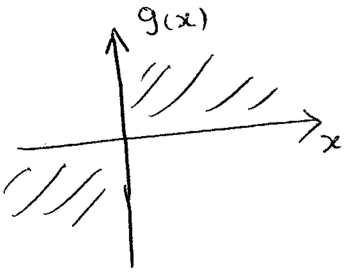
$V(0)=0$; $V(x)>0$, ~~$\forall x \setminus \{0\}$~~ $(\forall x \in D\setminus\{0\})$

$\dot{V}(x)$: negative definite

$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0$, $\forall x \in D\setminus\{0\}$

$\Rightarrow$ $\bar{x}=0$ locally assymptotically stable!

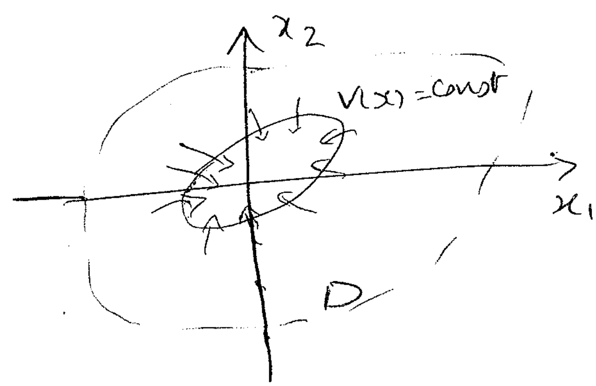
If $\dot{x} = -g(x)$



$V(x) = \int_0^x g(\xi)\, d\xi$

Sketch of the proof:

Key: everything follows from positive invariance of the set Ω_c:



$\Omega_c := \{x \mid V(x) \leq c\}$

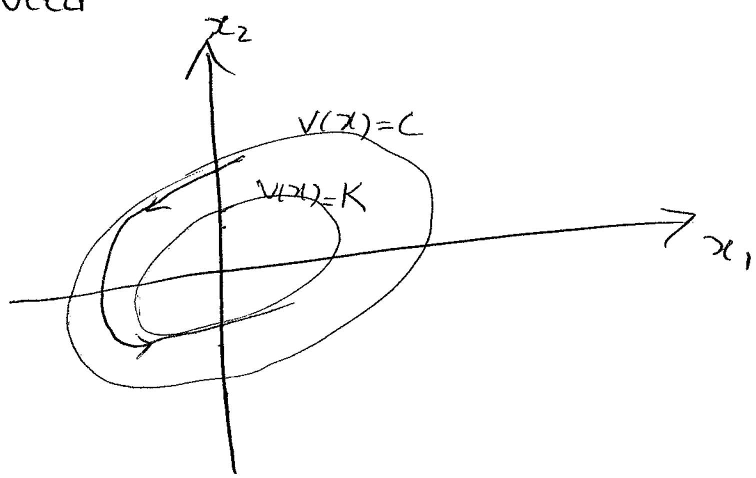
c ↑ Const

$\frac{dV}{dt} < 0 \quad , \forall x \in D \setminus \{0\}$

$V(t)$: non-increasing function of time
$V(t)$: bounded from below [by (zero)] $\Big\} \Rightarrow V(t)$: converges

$\Rightarrow \quad \boxed{\lim_{t \to \infty} V(t) = K}$

Need to show that $K = 0$, assume $K \neq 0$:



In $V(x) = C \rightarrow$ positive invariance!

Let $\boxed{\max \frac{dv}{dt} = -\gamma} < 0$

$\int_{t_0}^{t} \frac{dv}{dt} dt = V(x_t) - V(x_0) \quad \Rightarrow \quad$ (*) $\boxed{\begin{aligned} V(x(t)) &= V(x_0) + \int_{t_0}^{t} \frac{dv}{dt} dt \\ &\leq V(x_0) - \gamma(t - t_0) \end{aligned}}$

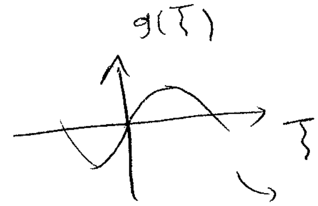
(*) $\Rightarrow$ there is $\underline{t}$ (large enough) s.t.
$V(x) < 0 \Rightarrow \boxed{K = 0}$ i.e. $\boxed{\lim_{t \to \infty} V(x(t)) = 0}$

Rest of the lecture:

- La Salle's invariance principle.
- Lyapunov theory for linear system.

Back to Ex:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a g(x_1) - b x_2 \quad ,$$
$$a>0,\ b>0$$



belongs to (I, III) at least locally!

$$\hookrightarrow V(x) = a\int_0^{x_1} g(\xi)\,d\xi + \frac{1}{2}x_2^2$$

$$\dot{V} = \frac{\partial V}{\partial x}\underbrace{f(x)}_{\dot{x}} = [a\,g(x_1) \mid x_2]\begin{bmatrix} x_2 \\ -a g(x_1) - b x_2 \end{bmatrix}$$

$$= \cancel{a g(x_1) x_2} - \cancel{a g(x_1) x_2} - b x_2^2 = 0\,x_1^2 - b x_2^2 \leq 0$$

$$= -[x_1 \quad x_2]\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightarrow \text{Negative semi-definite!}$$

$\Longrightarrow$ "Can't conclude local asymptotic stability from Lyapunov analysis!" "direct method"

(Just stable by oscillating but cannot tell it will stay at a final point)

La Salle's Invariance principle:

(applies to Time-Invariant systems)

$\dot{x} = f(x)$

provides conditions for asymptotic stability of $\bar{x} = 0$

even when $\dot{V}(x) \leq 0$!

Let $\Omega_c := \{x : V(x) \leq c\}$ be bounded, and let

$\dot{V}(x) \leq 0$ in Ω_c!

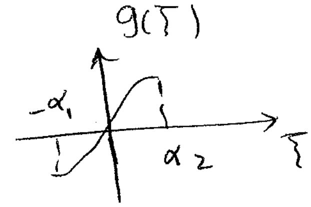
also, suppose that

$S := \{x \in \Omega_c ; \dot{V}(x) = 0\}$

and that M is the largest invariant set in S,

then $\lim_{t \to \infty} x(t) = M$, for all $x(0) \in \Omega_c$

Application to Ex:



$$S := \left\{ x \in \mathbb{R}^2 ; \begin{array}{l} x_1 \in [-\alpha_1, \alpha_2) \\ x_2 = 0 \end{array} \right\}$$

$x_2 \equiv 0 \rightarrow \dot{x}_2 \equiv 0 \Rightarrow \begin{array}{l} \dot{x}_1 = x_2 = 0 \\ \dot{x}_2 = -ag(x_1) - bx_2 \Rightarrow 0 = -ag(x_1) \end{array}$

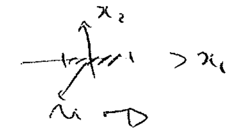
$\rightarrow \boxed{g(x_1) = 0} \Leftrightarrow \boxed{x_1 = 0}$

The only point in S that it can happens!

$\Rightarrow M = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ asymptotic stability of the origin!

M is a trivial set containing a single member only the equilibrium point $\bar{x}=0 \Rightarrow \lim_{t\to\infty} x(t) = 0 \Rightarrow \boxed{LAS}$

If $M \neq \{0\} \rightarrow$



$x_i \rightarrow$ perhaps stay in one of its points!

Back to linear systems:

$$\dot{x} = Ax \quad , A \in \mathbb{R}^{n\times n}$$

$$\boxed{\begin{aligned} V(x) &= x^T P x \\ P &= P^T > 0 \quad \left(\begin{matrix} \lambda_i(P) > 0 \\ (i=1,\dots,n) \end{matrix} \right) \end{aligned}}$$

$$\underset{\approx x^T x}{\lambda_{min}(P)} \leq x^T P x \leq \lambda_{max}(P)\, x^T x$$

"radially unbounded"

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} \underset{\dot{x}=Ax}{=} x^T (A^T P + PA)\, x = -x^T Q x$$

$$\dot{x} = Ax; \quad \underset{\substack{\Updownarrow \\ Re(\lambda_i(A)) < 0 \\ \text{for } i=1,\dots,n}}{A \text{ Hurwitz}} \quad \Leftrightarrow \quad \boxed{A^T P + PA = -Q}$$

[for any $Q = Q^T > 0$, there is $P = P^T > 0$ s.t.

$$A^T P + PA = -Q \quad !]$$