Lecture 1 March 04 / 2014

Last time

- existence & uniqueness of solutions

$$\dot{x} = f(x,t)$$

f: piecewise cts in t

Ex: $f(x) = x^{\frac{1}{3}} \Rightarrow$ non-unique solution

Continuity in x is not the point in x, we need (for uniqueness and existence)

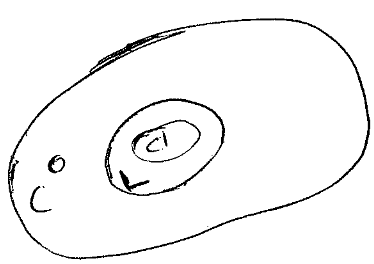
Lipschitz continuity!

$$\| f(x,t) - f(y,t) \| \leq L \| x-y \|$$

(global, local Lipschitz)

C^0: cts functions

C^1: ctsly diff ble



Ex 1: $f(x) = x^2$

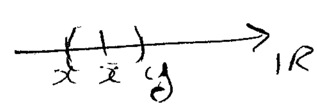
$$f(x) - f(y) = x^2 - y^2 = (x+y)(x-y)$$

$$|f(x) - f(y)| = |(x+y)(x-y)| = |x+y|\,|x-y|$$

Can we find a bound on $|x+y|$ - ?

yes (locally)

No (globally)



So, the function is locally Lipschitz but not globally!

there is no uniform bound (on $|x+y|$) !

Ex 2 : $f(x) = x^3$

$$f(x) - f(y) = x^3 - y^3 = (x^2 + xy + y^2)(x-y)$$

$\rightarrow$ There is no uniform bound on $|x^2 + xy + y^2|$!

Note

If $f(x)$ is ctsly (Continuously) differentiable ($f(x)$ is cts and $\frac{\partial f}{\partial x}$ is cts too)

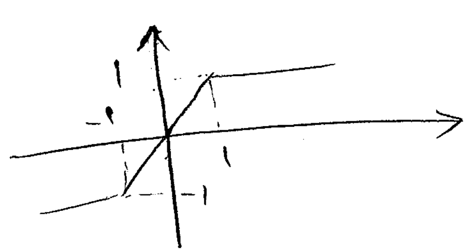
then is necessarily locally Lipschitz!

Ex 1 $\rightarrow$ $2x = \frac{\partial f}{\partial x}$ Ex 2 : $\frac{\partial f}{\partial x} = 3x^2$ $\rightarrow$ ctsly diffble

So, they are locally Lipschitz!

$\boxed{\text{Globally if } \frac{\partial f}{\partial x} \text{ is uniformly bounded!}}$

Ex 3 : $f(x) = \text{Sat}(x)$



$\rightarrow$ not differentiable yet globally Lipschitz!

Fact : let f be piecewise cts in time, If, in addition, f is : 1°) cts in x $\Rightarrow$ existence of Solutions on $[t_0, t_f]$

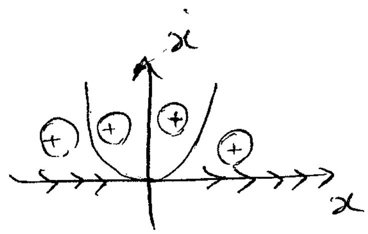
2°) locally Lipschitz $\Rightarrow$ existence and uniqueness on finite time interval. $[t_0, t_f]$

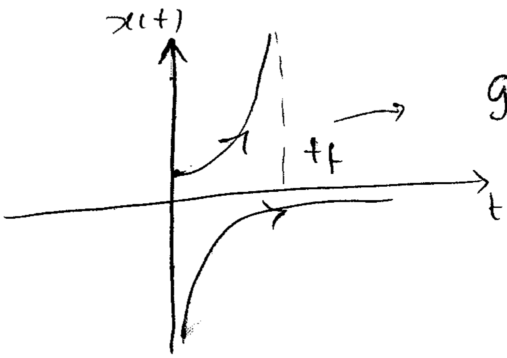
3°) Globally Lipschitz $\Rightarrow$ existence and uniqueness on $[t_0, \infty)$

$\Rightarrow$ These are sufficient conditions!

Back to Ex 1:

$f(x) = x^2 = \dot{x}$





going to blow up in finite time!

$$\begin{cases} x(t) = \dfrac{x_0}{1 - x_0 t} \\ t_f = \dfrac{1}{x_0}, \quad x_0 > 0 \end{cases}$$

Ex 4: $\dot{x} = -x^3$

Not globally Lipschitz yet Existence & uniqueness on $[0, \infty)$

Back to Linear Case:

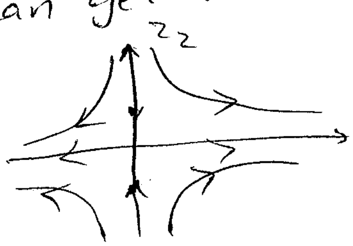
$$\dot{x} = A(t)\,x$$

$$\|f(x,t) - f(y,t)\| = \|A(t)(x-y)\| \leq \underbrace{\|A(t)\|}_{L} \|x-y\|$$

If it's piecewise continuous $\rightarrow L$
$\|A(t)\|$

$\Rightarrow$ If elements of $A(t)$ are piecewise cts function of time, then existence and uniqueness ✓

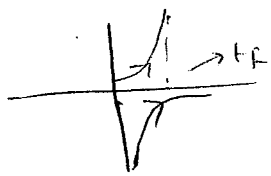
trajectories can get far from each other even in linear systems:



$$\begin{cases} \dot{z}_1 = z_1 \\ \dot{z}_2 = -z_2 \end{cases}$$

$z_1 \rightarrow$ How about finite horizon?

In Non linear



$\boxed{\text{Cts dependence on ICs}}$:

If f is locally Lipschitz (w.r.t. x) $\forall \varepsilon > 0$ and $T \in [0, t_f]$, there is $\delta = \delta(\varepsilon, T)$ s.t.

$$\|x_0 - y_0\| < \delta \Rightarrow \|\varphi(x_0, t) - \varphi(y_0, t)\| < \varepsilon$$

$\boxed{\text{for all } 0 \leq t \leq T}$

How about dependence w.r.t parameters?

$\dot{x} = f(x, t, \mu)$
$\quad\hookrightarrow$ Const. vector of parameters

Add : $\dot{\mu} = 0$

$z = \begin{bmatrix} x \\ \mu \end{bmatrix} \Rightarrow \dot{z} = g(z, t)$, $g := \begin{bmatrix} f \\ 0 \end{bmatrix}$

Sensitivity of Solutions w.r.t. parameters :

$\dot{x} = f(x, t, \mu)$
f is cts diffble in the vicinity of $\bar{\mu}$:

$$x(t,\mu) = x_0 + \int_{t_0}^{t} f(x, \tau, \mu)\, d\tau$$

$$\frac{\partial x}{\partial \mu}(t,\mu) = 0 + \int_{t_0}^{t} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial \mu} + \frac{\partial f}{\partial \mu} \right) d\tau$$

Notation : $x_\mu = \frac{\partial x}{\partial \mu}$, $\Rightarrow x_\mu(t,\mu) = \int_{t_0}^{t} \left(\frac{\partial f}{\partial x} x_\mu + \frac{\partial f}{\partial \mu} \right) d\tau \Big| \frac{d}{dt}$

$$\boxed{\dot{x}_\mu(t,\mu) = \frac{\partial f(x,t,\mu)}{\partial x} x_\mu(t,\mu) + \frac{\partial f(x,t,\mu)}{\partial \mu}}$$

Ex : If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \Rightarrow$

$$x_\mu = \frac{\partial x}{\partial \mu} = \begin{bmatrix} x_{1\mu_1} & x_{1\mu_2} & x_{1\mu_3} \\ x_{2\mu_1} & \underset{\frac{\partial x_2}{\partial \mu_2}}{x_{2\mu_2}} & x_{2\mu_3} \end{bmatrix} \rightarrow \text{Sensitivity Matrix}$$

Given $x(t, \bar{\mu})$

$$x(t, \mu) = x(t, \bar{\mu}) + \boxed{\left.\frac{\partial x}{\partial \mu}\right|_{\bar{\mu}}} (\mu - \bar{\mu}) + \text{H.O.T.}$$

it's important to see how solution changes by changing μ !