Lecture 08 Feb. 25

Last time:

- Hopf Bifurcation

  - Super } critical ☺ ☹
  - Sub

- Scaling / Non-dimensionalization

Today:

- Center Manifold theory

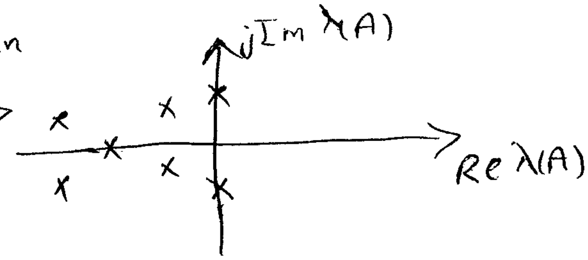
  (Khalil: Chapter 8)

$\dot{x} = f(x)$ (1) ; $x(t) \in \mathbb{R}^n$

with an eq. point $\bar{x}$ @ the origin!

$\rightarrow f(\bar{x}) = 0 \rightarrow \bar{x} = 0$

$$A := \frac{\partial f}{\partial x}\Big|_{\bar{x}=0} \overset{\in \mathbb{R}^{n\times n}}{\Longrightarrow}$$



K e_value @ $j\omega$_axis
(stability boundary)

n−K e_values in the LHP ($\mathrm{Re}\,\lambda(A) < 0$)

tough lock with using linearization to figure Stability of the eq. point ($\bar{x}=0$) of nonlinear system $\boxed{1}$

Ex: $\dot{x} = a x^3$

$A \cong 0$ yet $\underbrace{\quad}_{\bar{x}=0}$ (globally) assymptotically stability $\to$ assymptotically stable (globally), $a<0$

unstable, $a>0$

Need to examine the role of nonlinear terms!

Rewrite 1 as: $(\dot{x} = f(x))$

$\dot{x} = Ax + \tilde{f}(x)$ [Taylor series of $f(x)$ around $\bar{x}=0$]

$$\Rightarrow f(x) = \underbrace{\cancel{f(0)}}_{0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} x}_{A.x} + \underbrace{H.O.T}_{\tilde{f}(x)}$$

properties of $\tilde{f}$:

1° $\tilde{f}(0)=0$ (because $f(0) = 0 = A.0 + \tilde{f}(0) \to \boxed{\tilde{f}(0)=0}$)

2° $\left.\frac{\partial \tilde{f}}{\partial x}\right|_{\bar{x}=0} = 0$ (because $\left(\cancel{\frac{\partial f}{\partial x}} = \cancel{\left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}} + \frac{\partial \tilde{f}}{\partial x}\right)\Big|_{\bar{x}=0} \to \boxed{\left.\frac{\partial \tilde{f}}{\partial x}\right|_{\bar{x}=0} = 0}$)

So far:

we re-wrote (1):

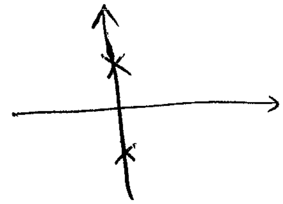
$\dot{x} = Ax + \tilde{f}(x)$, (2)

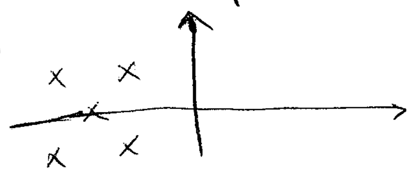
where $A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0}$ and 1° $\tilde{f}(0) = 0$ & 2° $\left.\frac{\partial \tilde{f}}{\partial x}\right|_{\bar{x}=0} = 0$

Change of coordinates:

$$\begin{array}{l} \mathbb{R}^k \leftarrow \\ \mathbb{R}^{n-k} \leftarrow \end{array} \begin{bmatrix} y \\ z \end{bmatrix} = \underbrace{T}_{\text{fixed matrix}} x$$

(3) $$\begin{cases} \dot{y} = A_1 y + g_1(y,z) & \text{where } \lambda(A_1) \\ \dot{z} = A_2 z + g_2(y,z) & \text{where } \lambda(A_2) \end{cases}$$





(If the system was linear $g_1 = g_2 = 0$ but now they are coupled)

Ex: 1-D bifurcations in higher dimensions

$$\begin{cases} \dot{y}_1 = 0 \cdot y_1 + \tilde{g}_1(y_1, \underbrace{\alpha}_{\text{bifurcation parameter}}, z) \\ \dot{\alpha} = 0 \end{cases}$$

$$\dot{z} = A_2 z + g_2(y_1, \alpha, z)$$

$y_1(t) \in \mathbb{R}^1 \rightarrow y = \begin{bmatrix} y_1 \\ \alpha \end{bmatrix}, \quad g_1 = \begin{bmatrix} \tilde{g}_1 \\ 0 \end{bmatrix}$

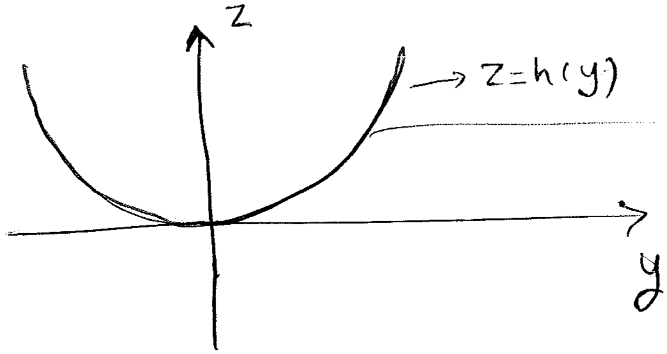
$$\rightarrow A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \boxed{\begin{cases} g_i(0,0) = 0 \quad i = 1,2 \\ \left.\dfrac{\partial g_i}{\partial y}\right|_0 = 0 \\ \left.\dfrac{\partial g_i}{\partial z}\right|_0 = 0 \end{cases}}$$

Fact (Thm)

There is an invariant manifold $z = h(y)$ in the neighborhood of the origin that satisfies $h(0) = 0$, $\left.\dfrac{\partial h}{\partial y}\right|_0 = 0$!

Invariant: you start there $\Rightarrow$ you stay there (for all times)

$z(0) = h(y(0)) \Rightarrow z(t) = h(y(t))$; for all $t \geq 0$



if we start here we stay here!
(Q: if start from somewhere else, what will happen?)

$$T^{-1} = S \rightarrow \left.\begin{array}{l} x = T^{-1}\begin{bmatrix} y \\ z \end{bmatrix} \\ \dot{x} = Ax + \tilde{f}(x) \end{array}\right\} \rightarrow T^{-1}\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = AT^{-1}\begin{bmatrix} y \\ z \end{bmatrix} + \tilde{f}(x)$$

hit with T $\rightarrow$ from both sides

$$\boxed{\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \underbrace{TAT^{-1}}_{\left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]}\begin{bmatrix} y \\ z \end{bmatrix} + T\tilde{f}(x)} \rightarrow \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$\rightarrow$ "If the manifold is closed it's a periodic orbit"

Main result:

If the origin of the reduced system:

$$\dot{y} = Ay + g_1(y, h(y)) \quad \left\{\begin{array}{l} z = h(y) \\ h(0) = 0 \\ \left.\frac{\partial h}{\partial y}\right|_0 = 0 \end{array}\right.$$

is asymptotically stable, (respectively unstable) then, the origin of (2) [full system] is asymptotically stable (or unstable)!

Question: what conditions does $h(y)$ have to satisfied (i.e. how to find $h(y)$)

Introduce:

$$\omega := z - h(y)$$

$$\omega = 0 \Rightarrow \boxed{\dot{\omega} = 0}$$ (because if I start there I stay there)

$$\dot{\omega} = \dot{z} - \frac{\partial h}{\partial y}\dot{y}$$

from (3) in page 3 $\rightarrow$ $$\dot{\omega} = A_2 z + g_2(y,z) - \frac{\partial h}{\partial y}(A_1 y + g_1(y,z))$$

$z = h(y)$ $\rightarrow$ (*) $$\boxed{\dot{\omega} = A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}(A_1 y + g_1(y, h(y))) = 0}$$

$\rightarrow$ By solving this DE we can find $h(y)$! ("hard to solve")

y is independent, $h(y)$ is dependent!

Ex: $y(t) \in \mathbb{R}^2$, $z(t) \in \mathbb{R}^3$ $\Rightarrow$ $z = h(y) = \begin{bmatrix} h_1(y) \\ h_2(y) \\ h_3(y) \end{bmatrix}$

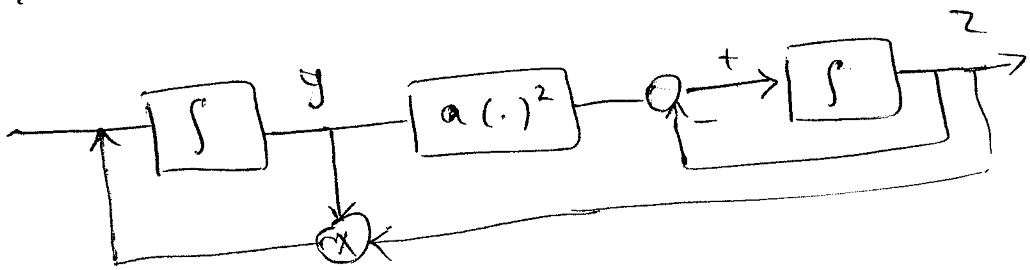
In general, finding solutions to (*) is difficult!

(even if y & $z \in \mathbb{R}$ it's difficult to solve)

$\boxed{\text{(*) characterizes center manifold } h(y)}$

Ex : $\begin{cases} \dot{y} = 0 \cdot y + yz & ; \ y(t) \in \mathbb{R} \\ \dot{z} = -z + ay^2 & ; \ z(t) \in \mathbb{R} \end{cases}$

$A_1 = 0$, $A_2 = -I$, $g_1 = yz$, $g_2 = ay^2$



$\circledast$: $\quad -h + ay^2 - \frac{\partial h}{\partial y} \, y \, h(y) = 0$

$$\begin{cases} yh \dfrac{dh}{dy} = ay^2 - h \\ h(0) = 0 \ ; \ \dfrac{\partial h}{\partial y}\Big|_0 = 0 \end{cases}$$

Taylor series of $h(y)$ around $y = 0$

$$h(y) = \underbrace{h(0) + \frac{\partial h}{\partial y}\Big|_0 \, y}_{0} + h_2 y^2 + h_3 y^3 + \dots$$

(what is locally happening?)