Last time:

- Application of Bendixon ThM
  [lack of periodic orbits for 2nd order Systems]

- positively invariant sets
  $x(t_0) \in M \Rightarrow x(t) \in M$, for all $t \geq t_0$

Today:

- Poincare - Bendixon thm

- Hopf bifurcation

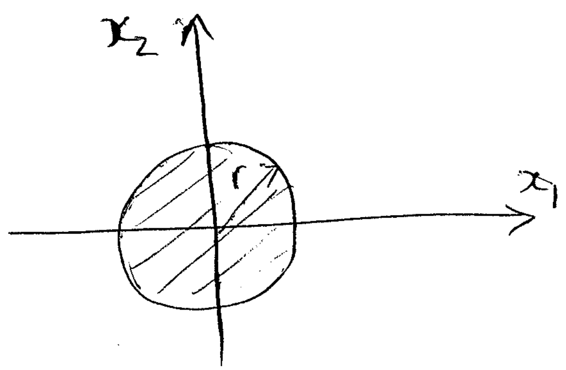
- Sub / - super } critical

Last time (Example)

$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$

$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$

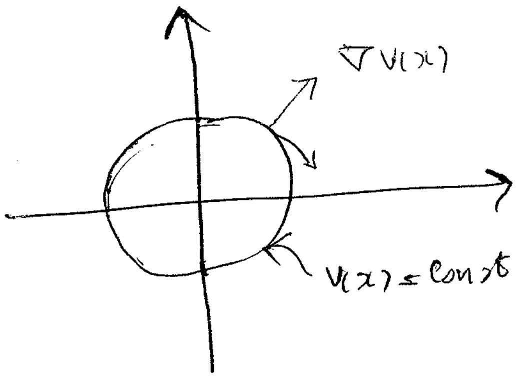
$B_r = \{x \in \mathbb{R}, x_1^2 + x_2^2 \leq r^2\}$ is positively invariant for big enough r!

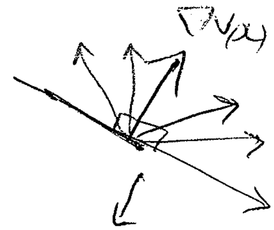


If r is big enough $\rightarrow$
B_r is positive invariant!

$V(x) = \text{Const}$

$$= 2(x_1 f_1 + x_2 f_2) \cdots \text{If} \xrightarrow{\leq 0} \boxed{r \geq \sqrt{\tfrac{3}{2}}}$$





If f is in this space the inner product will be positive but we want to make sure that the angle between f & ∇V is bigger than 90°!

Poincare - Bendixon Thm:

2nd order System:

Given
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

and a set M which is Closed and bounded, then if

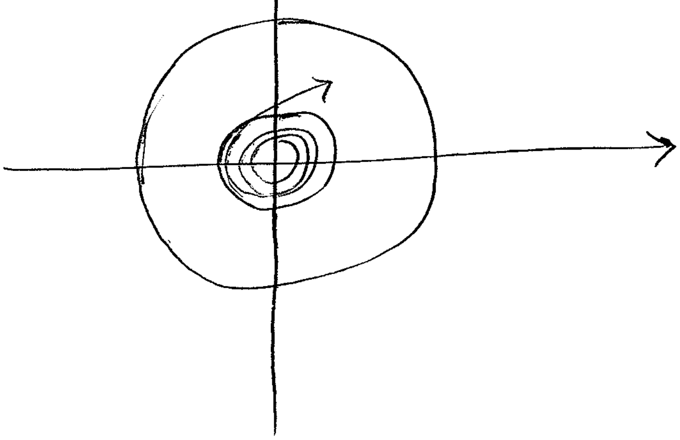
(a) there are no eq. points in M

(b) M is positively invariant

Then, $\boxed{M \text{ Contains a periodic orbit!}}$

Note: It can be shown that The above Thm holds if M contains a single eq. point which is either unstable node or unstable focus! (both eigenvalues sit in RHP)

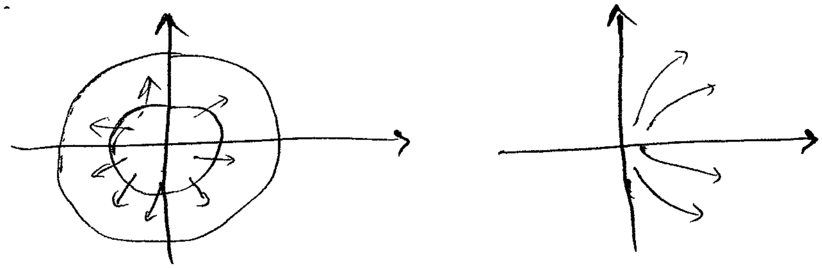
②



$\lambda_1, \lambda_2 \in \mathbb{C}$
with $\left.\begin{matrix} Re(\lambda_1) \\ Re(\lambda_2) \end{matrix}\right\} > 0$
(focus)

unstable node:

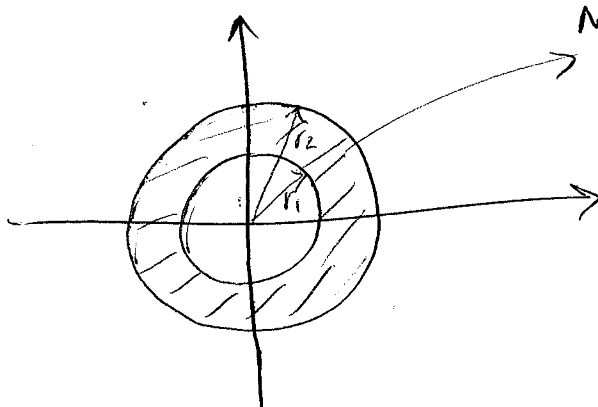
$\lambda_1, \lambda_2 > 0$
real



Ex1: $\dot{x}_1 = -x_2$
$\dot{x}_2 = x_1$ $\Rightarrow$ $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\lambda_{12} = 0 \pm j$

$M = \{x \in \mathbb{R}^2;\ r_1^2 \leq x_1^2 + x_2^2 \leq r_2^2\}$



$\Rightarrow$ $f^T(x)\nabla V(x) = \begin{bmatrix} -x_2 & +x_1 \end{bmatrix} \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = -2x_1x_2 + 2x_1x_2 = 0$

$$\left.\frac{\partial f}{\partial x}\right|_{\bar{x}=0} = \begin{bmatrix} 1 & +1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ → Compute eigen values:

$\det(SI - A) = (S-1)(S-1) + 2 = 0 \Rightarrow S^2 - 2S + 3 = 0$ →

$$\lambda_{12} = \frac{2 \pm \sqrt{-8}}{2}$$

→ So, they are unstable focus → $\begin{cases} \alpha \pm j\beta \\ \alpha > 0 \end{cases}$

→ according to poincare-Bendixon there has to be a periodic orbit in it!

Is there any other eq. point? $f_1(\bar{x}_1, \bar{x}_2) = 0$, $f_2(\bar{x}_1, \bar{x}_2) = 0$

→ $\bar{x}_1 + \bar{x}_2 - (\bar{x}_1^2 + \bar{x}_2^2)\bar{x}_1 = 0$ (*) → $\bar{x}_1 = 0 \rightarrow \bar{x}_2 = 0$

So, we can devide (*) by $\bar{x}_1$ → $\boxed{\dfrac{\bar{x}_1 + \bar{x}_2}{\bar{x}_1} = \bar{x}_1^2 + \bar{x}_2^2}$

$f_2 = 0 \Rightarrow -2\bar{x}_1 + \bar{x}_2 = \bar{x}_2 \dfrac{\bar{x}_1 + \bar{x}_2}{\bar{x}_1} = \bar{x}_2(\bar{x}_1^2 + \bar{x}_2^2)$

→ $-2\bar{x}_1^2 + \bar{x}_1\bar{x}_2 = \bar{x}_1\bar{x}_2 + \bar{x}_2^2$ → $\boxed{0 = 2\bar{x}_1^2 + \bar{x}_2^2}$ → $\boxed{\bar{x}_1 = \bar{x}_2 = 0}$

So, we have a single eq. point!

Back to Bifurcations: (Hopf)

So far:

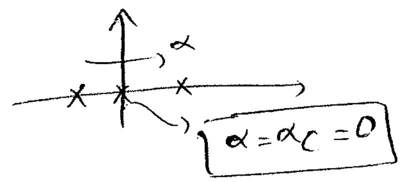
1° Fold, $\dot{x} = \alpha \pm x^2$

2° transcritical, $\dot{x} = \alpha x \pm x^2$

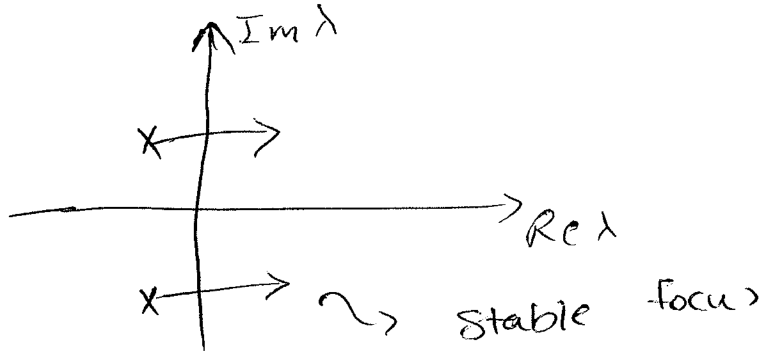
3° pitch fork, $\dot{x} = \alpha x \pm x^3$

- $A_c = \left.\frac{\partial f}{\partial x}\right|_{\alpha=\alpha_c} = 0$ (at critical value of parameter linearization cannot tell anything and when we are in 1D it should be zero (on j-axis))

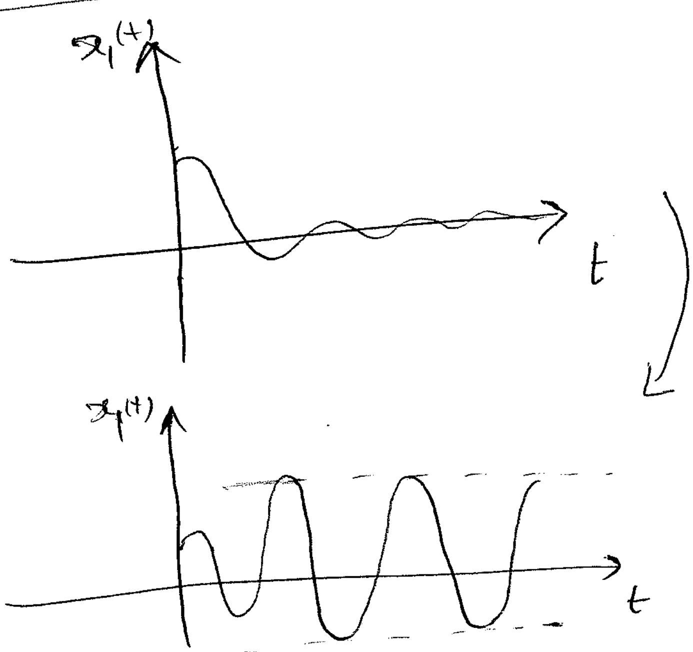
Critical value of α for pitchfork $\rightarrow$ $\dot{x} = \alpha x \pm x^3$



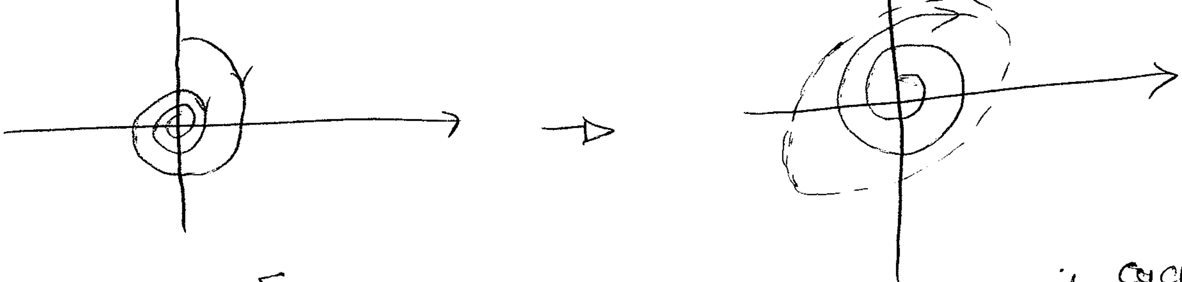
we have not discussed this following example yet:



Time dependence:



when it goes to RHP!



Stable Focus $\rightarrow$ Stable - limit cycle

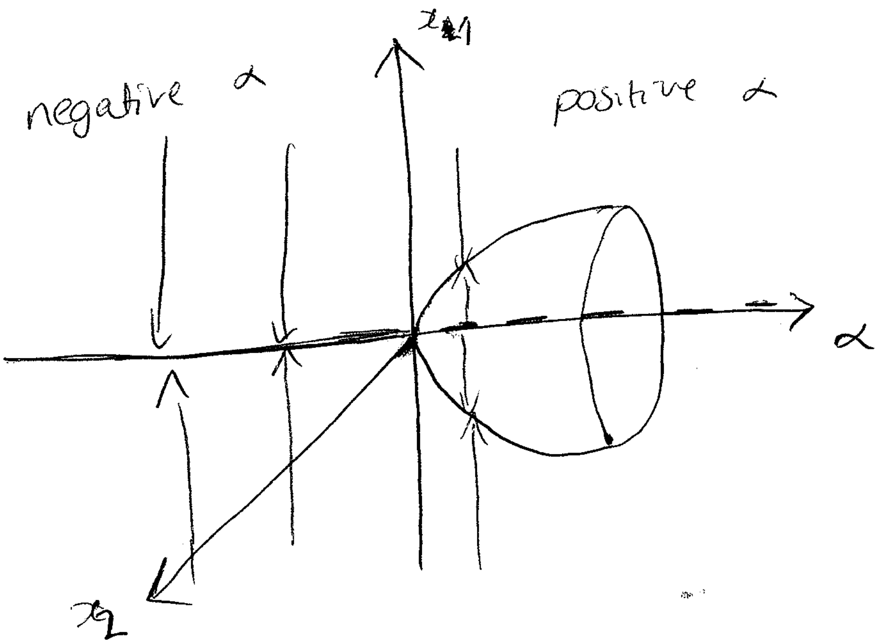
<u>EX</u>:

$$\dot{x}_1 = x_1(\alpha - x_1^2 - x_2^2) - x_2$$
$$\dot{x}_2 = x_2(\alpha - x_1^2 - x_2^2) + x_1$$

$\rightarrow$ polar coordinate $\rightarrow$

$$\left.\begin{array}{l} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{array}\right\} \rightarrow \boxed{\begin{array}{l} \dot{r} = \alpha r - r^3 \\ \dot{\theta} = 1 \end{array}}$$

$$\bar{r}(\alpha - \bar{r}^2) = 0 \rightarrow \bar{r} = 0 \quad \text{or} \quad \bar{r} = \sqrt{\alpha} \text{ if } \alpha > 0$$



$\rightarrow$ $\boxed{\text{Super - critical}}$ ☺

will not go very far from the origin!