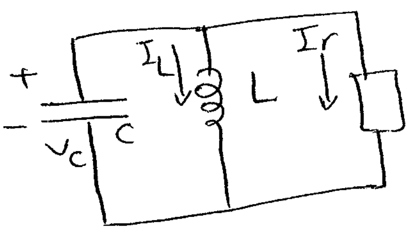
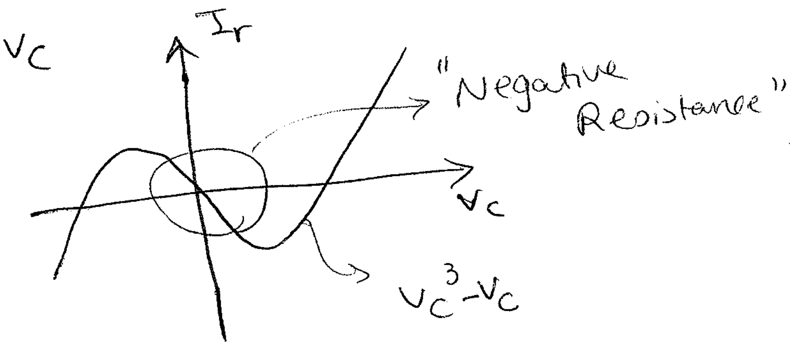
Lecture 03

Last Time:

* Non-linear phenamena
  - Finite escape time
    $\dot{x} = x^2$
  - Multiple isolated eq. V.S. Null(A) in $\dot{x} = Ax$!
    $\dot{x} = x^2 - 1$
  - Limit cycles
    Ex: Van der pol oscillator



$I_r = I_r(V_C) = V_C^3 - V_C$



$$\begin{cases} \dot{I}_L = \frac{1}{L} V_C \\ \dot{V}_C = -\frac{1}{C} I_L + \frac{1}{C}(V_C - V_C^3) \end{cases}$$

unique eq. point:

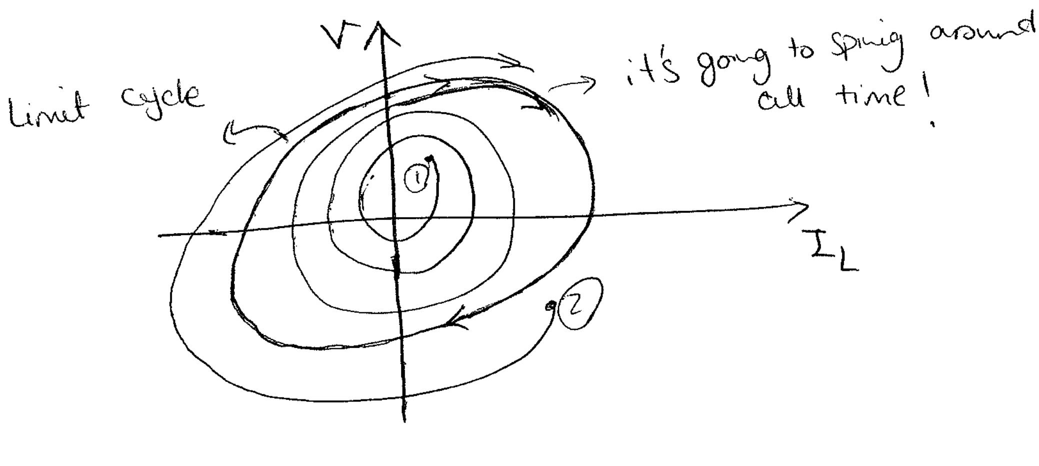
$\begin{bmatrix} I_L \\ V_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ unstable

because $\rightarrow A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & +\frac{1}{C} \end{bmatrix}$

$\det(SI - A) = S^2 - \frac{1}{C} S + \frac{1}{LC}$

Today:

- Chaos
- Bifurcations



$V_C^3 \rightarrow$ Stabilizing term $\neq$ negative resistance

$\Downarrow$ Limit spinging to the cycle and prevent blowing up!

negative resistance $\downarrow$ dsstablizing $\rightarrow$ departing from the origin

_"chaos"

No universally accepted definition characterized by

- HUGE sensitivity w.r.t initial conditions
- A periodic "long term behavior" ($t \to \infty$ trajectories don't go to eq. points or limit cycles.) $\rightarrow$ "<u>strange</u> attractors"

Ex: <u>lorentz attractor (system):</u>

<u>3rd order system:</u> {(you need go to third dimension to have chaotic behavior in CT.) (or 2 dim with input)

$$\left.\begin{aligned} \dot{x} &= a(x-y) \\ \dot{y} &= x(b-z)-y \\ \dot{z} &= xy - cz \end{aligned}\right\} \Rightarrow$$

"Simplified model of convective rolls in athmosphere"

a, b, c: Constant parameters. $\left[a=10, \; b=28, \; c=\frac{8}{3}\right]$ (chaos)

Linearization: $A = \left[\begin{array}{cc|c} a & -a & 0 \\ b & -1 & 0 \\ \hline 0 & 0 & -c \end{array}\right] \rightarrow eig = U\left[(-c), \; eig\begin{bmatrix} a & -a \\ b & -1 \end{bmatrix}\right]$

$$sI - \begin{bmatrix} a & -a \\ b & -1 \end{bmatrix} = \begin{bmatrix} s-a & a \\ -b & s+1 \end{bmatrix} \xrightarrow{det} \Rightarrow (s-a)(s+1)+ab=0$$

$$s^2 - (a-1)s + ab - a \Rightarrow \begin{cases} \text{For stability we should have;} \\ ab - a > 0 \\ -(a-1) > 0 \end{cases}$$

for $a > 1$ is unstable

Conclusion:

If $a, b, c > 0$: origin stable $\Longleftrightarrow$ $\boxed{\begin{matrix} a<1 \\ b>1 \end{matrix}}$

Bifurcations:

Back to first order systems:

$$\frac{dx}{dt} = f(x, \underbrace{\alpha}_{\text{parameter}}) \quad , \alpha \in \mathbb{R}$$

$x(t) \in \mathbb{R}$: Scalar

"what makes studying of behavior for first order systems interesting is a parameter's existance!"

Bifurcation: Change in qulitative behavior as parameter changes $\Rightarrow$

- creation or dissapearance of eq. points
- change of their stability properties

There are three classes of 1st order bifurcation:

1° Fold $\quad \dot{x} = \alpha \pm x^2$

2° Transcritical $\quad \dot{x} = \alpha x \pm x^2$

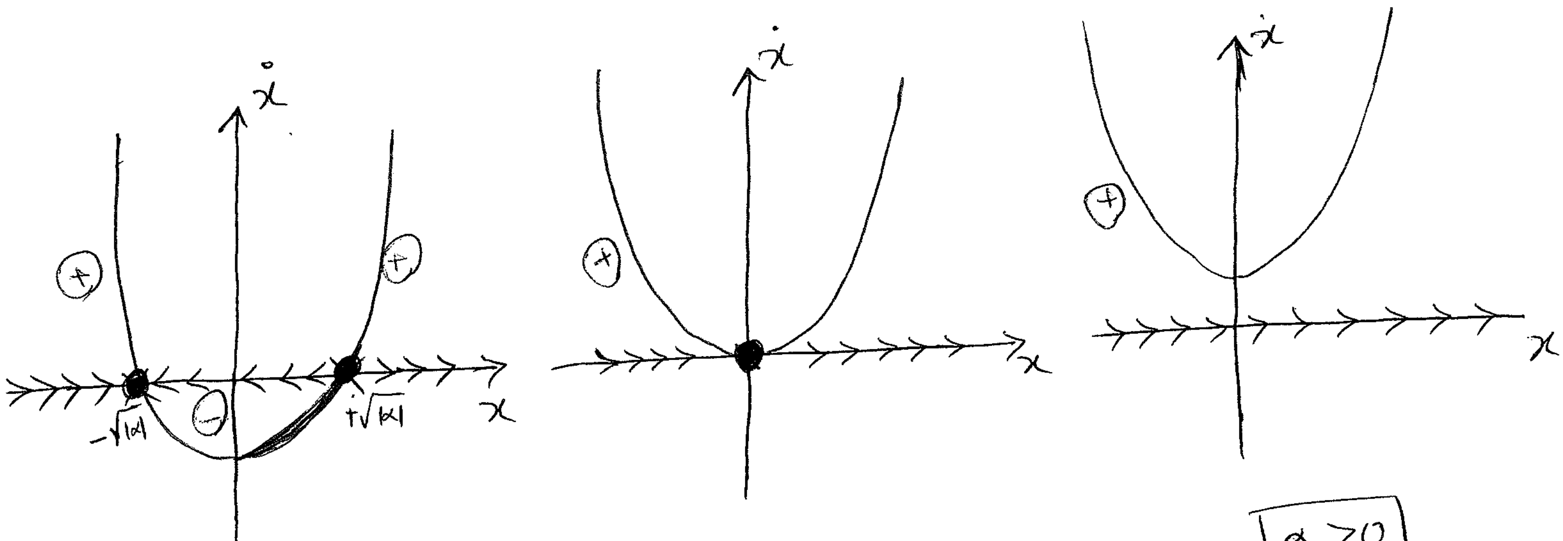
3° pitchfork $\quad \dot{x} = \alpha x \pm x^3$

1° Fold (Saddle node "blue sky")

Ex: $\dot{x} = \alpha + x^2$

Vary $\alpha \in (-\infty, +\infty)$

Eq. points: $0 = \alpha + \bar{x}^2 \Rightarrow \bar{x}^2 = -\alpha \Rightarrow \bar{x} = \begin{cases} \pm\sqrt{|\alpha|}, & \alpha \leq 0 \\ \text{none}, & \alpha > 0 \end{cases}$



$\boxed{\alpha < 0}$ $\boxed{\alpha = 0}$ $\boxed{\alpha > 0}$

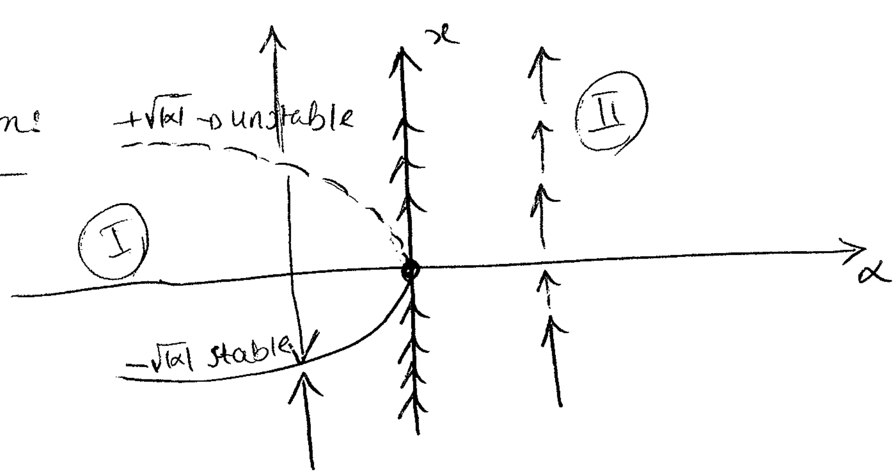
Critical value: $\boxed{\alpha_c = 0}$

transition from two eq. points to one and then to none!

Linearization: $\frac{\partial f}{\partial x} = 2x \rightarrow \left|\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{cases} \pm 2\sqrt{|\alpha|}, & \alpha < 0 \\ 0, & \alpha = 0 \\ \text{none}; & \alpha > 0 \end{cases}$

Note @ $\alpha = \alpha_c \Rightarrow$ linearization doesn't offer useful info!

Bifurcation diagram:



I $\rightarrow$ x decreases!

II $\rightarrow$ x increases!