lecture 02

Jan. 23 /2014

Last time:

- Non Linear Systems

$$\frac{dx}{dt} = f(\underset{\text{state}}{x}, \underset{\text{input}}{u}, \underset{\text{time}}{t})$$

Ex: $\dot{x} = \sin x$

Eq. points: $f(\bar{x}) = 0$

$\bar{x} = k\pi$, $k = 0, \pm 1, \pm 2, \dots$

Linearization: $x(t) = \underset{\text{e.P}}{\bar{x}} + \underset{\text{perturbation}}{\tilde{x}(t)}$

$$\dot{\tilde{x}} = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} \cdot \tilde{x}$$

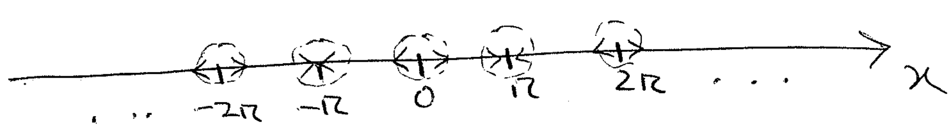
Jacobian: $A = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \cos \bar{x} = \cos k\pi = (-1)^k$

$$\Rightarrow A = \begin{cases} +1 , & k: \text{even} \\ -1 , & k: \text{odd} \end{cases}$$

Linearized dynamics (around $\bar{x} = k\pi$) $\rightarrow$

$$\dot{\tilde{x}} = \begin{cases} +\tilde{x} , & k: \text{even} \\ -\tilde{x} , & k: \text{odd} \end{cases} \rightarrow \tilde{x}(t) = \begin{cases} e^{t} , & k: \text{even} \\ e^{-t} , & k: \text{odd} \end{cases}$$

if you start with k: even then you will go far from $\bar{x}$ exponentially!



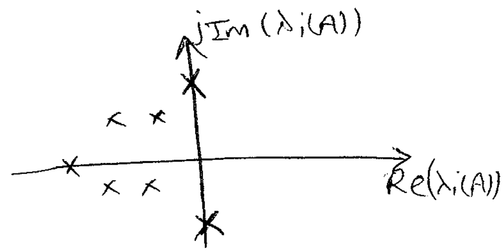
Linearization can be used to assert local Stability properties of eq. points of a nonlinear systems!

(a) $\left\{ \begin{array}{l} Re(\lambda_i(A)) < 0 \\ \text{for } i=1,\dots,n \\ A \in \mathbb{R}^{n\times n} \\ \quad \to \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} \end{array} \right.$ $\Rightarrow$ $\bar{x}$: eq. point of $\dot{x} = f(x)$ is locally asymptotically stable! ($\lim_{t\to\infty} x(t) = \bar{x}$ for all $x(t_0)$ "close" to $\bar{x}$) (+) [you stay close to $\bar{x}$ for all t]

(b) If there is $i \in \{1,\dots,n\}$ with $Re(\lambda_i(A)) > 0 \Rightarrow$ unstable

Issues: (1) Can't say anything about global properties!

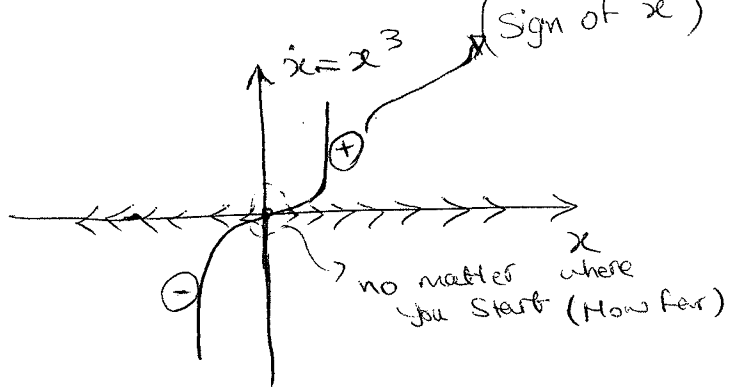
(2) $Re(\lambda_i(A)) \leq 0$ for all i! but there are some of them with $Re(\lambda_i(A)) = 0$ (Inconclusive)

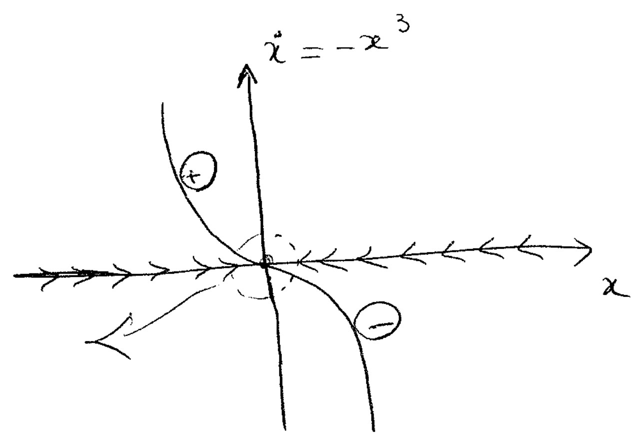


How far from an eq. point can we start for converging?

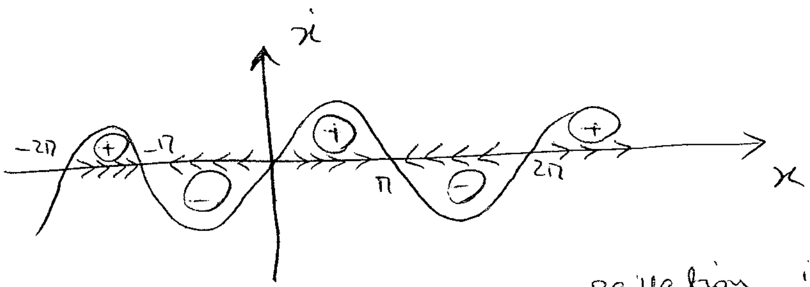
Ex: $\dot{x} = x^3$ or $\dot{x} = -x^3$ $\xrightarrow{\text{Linearization}}$ $\left\{ \begin{array}{l} \dot{\tilde{x}} = 0 \cdot \tilde{x} \\ \tilde{x}(t) = \tilde{x}(0) \end{array} \right.$

$$\frac{\partial f}{\partial x} = \pm 3x^2 \Big|_{\bar{x}=0} = 0$$



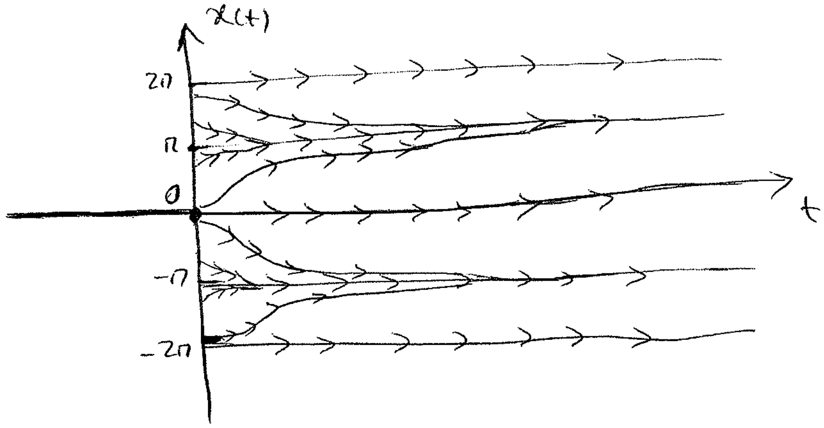


Back to: $\dot{x} = \sin x$

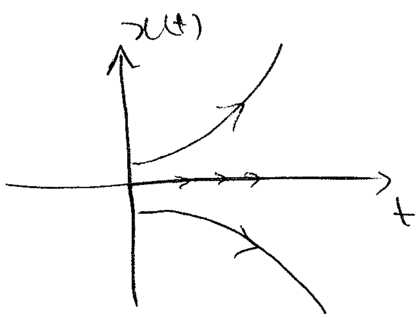


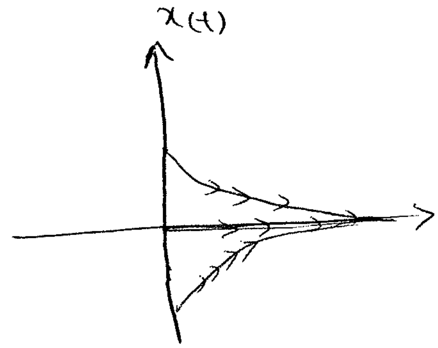
There cannot be any oscillation in first order systems!

It should decrease or increase monotically!



For the previous Example:





Range of "nonlinear phenomena"

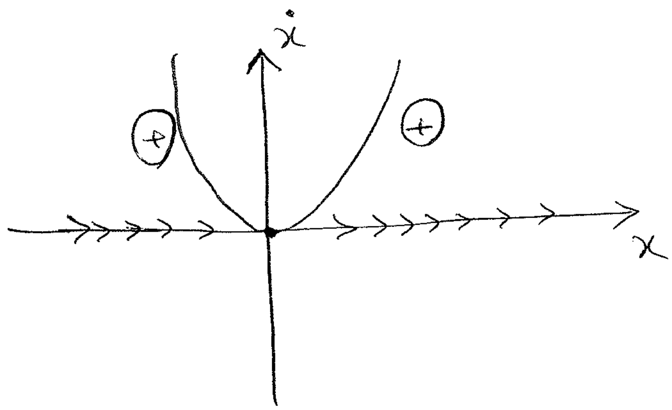
1°) Finite "escape" time (this can never happen in linear finite dimensional systems)

$x(t) \xrightarrow{t \to T} +\infty$
(even for finite t)

$\dot{x}(t) = Ax \to x(t) = e^{At}x(0)$

(If there is i s.t. $Re(\lambda_i(A)) > 0$) $\Rightarrow$ $\|x(t)\| \xrightarrow{t \to \infty} +\infty$

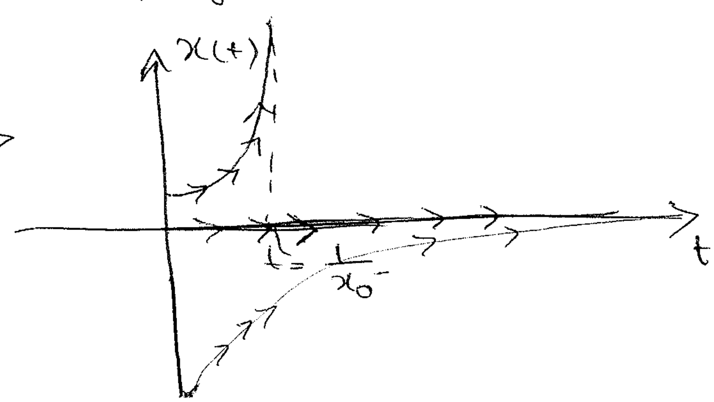
Ex: $\dot{x} = x^2$ $\rightarrow$ Linearization : $\dot{\tilde{x}} = 0 \quad \tilde{x} = 0$



$$\int_{x_0}^{x(t)} x^{-2} dx = \int_{t_0}^{t} dt \Rightarrow -\frac{1}{x}\Big|_{x_0}^{x(t)} = t - t_0$$

$$\rightarrow \frac{1}{x_0} - \frac{1}{x(t)} = t - t_0 \Rightarrow x(t) = \frac{x_0}{1 - x_0(t - t_0)}$$

Set $t_0 = 0$ $\longrightarrow$ $\boxed{x(t) = \dfrac{x_0}{1 - x_0 t}}$ $\Rightarrow$



2°) Multiple Isolated equilibria:

Ex: $\dot{x} = x^2 - 1$ $\longrightarrow$ $\bar{x}^2 - 1 = 0 \rightarrow \bar{x} = \pm 1$

Ex: Logistic Equations :

Naive (Linear) model of population growth

$\dot{x} = \alpha x$, $\alpha > 0$

$x(t) = e^{\alpha t} x(0)$ $\rightarrow$ (Not reasonable)

problem: $\dfrac{\dot{x}}{x} = \alpha = \text{const} \Rightarrow$ (normalized growth rate is const) but it should be a decaying function!

More reasonable model (decaying function):

$$\frac{\dot{x}}{x} = \text{decaying function of } x$$

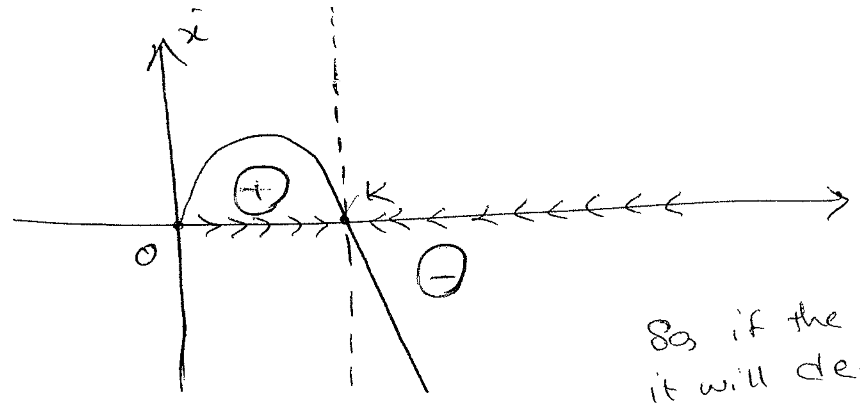
Simplest model: logistic eq.

$$\frac{\dot{x}}{x} = \alpha\left(1 - \frac{x}{K}\right); \quad \alpha, K: \text{positive constants} \quad \alpha, K > 0$$

K: carrying capacity

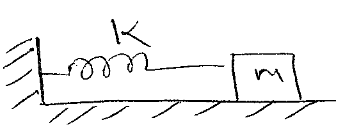
$$\Rightarrow \dot{x} = \alpha\left(1 - \frac{x}{K}\right)x \rightarrow \text{Eq. points: } \begin{cases} \bar{x} = 0 \\ \bar{x} = K \end{cases}$$

population $(x) > 0$

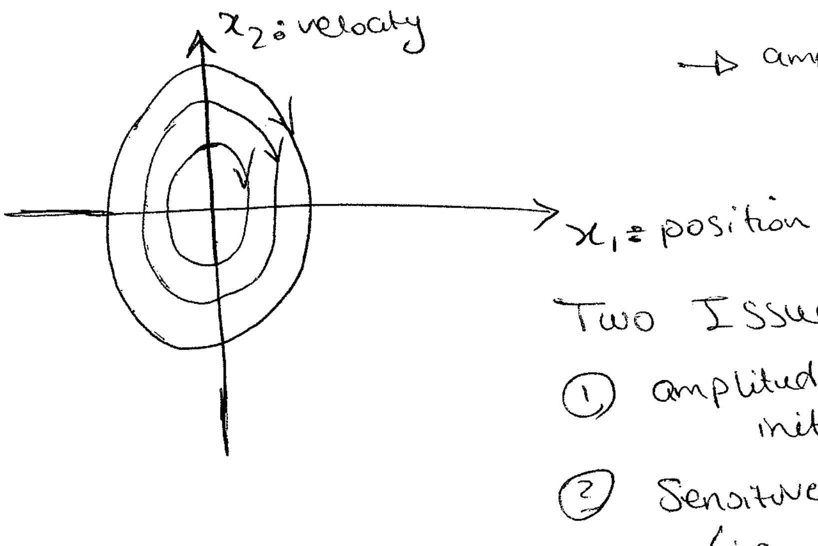


So if the population is greater than K, it will decrease and if it's less than K, it will increase!

3°) Limit cycles (i.e. sustained oscillations)
robust



$$\ddot{x} + \frac{K}{m}x = 0 \quad \text{(Harmonic oscillator)}$$

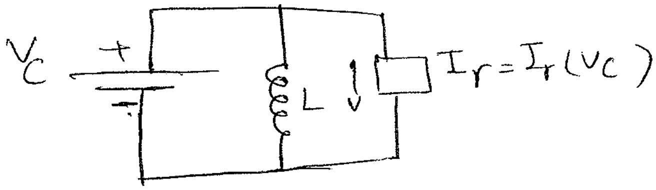


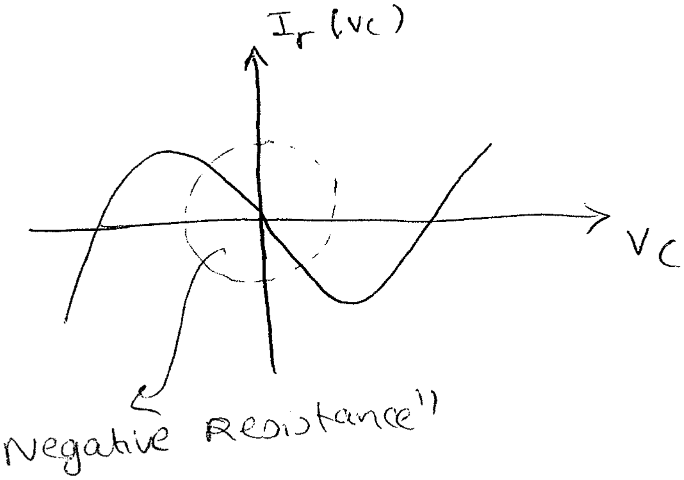
→ amplitude is going to change with initial conditions!

Two Issues:

1. amplitude of oscillations depends on the initial conditions!
2. Sensitive to modeling imperfections (i.e., non robust)

van der pol oscillator:





$$\begin{cases} \dot{I}_L = \frac{1}{L} v_c \\ \dot{v}_c = -\frac{1}{c} I_L + \frac{1}{c}(v_c - v_c^3) \\ I_r(v_c) = -v_c + v_c^3 \end{cases}$$

unique Eq. point $\rightarrow \begin{bmatrix} \bar{I}_L \\ \bar{v}_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Linearization: $\begin{bmatrix} \dot{\tilde{I}}_L \\ \dot{\tilde{v}}_c \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{c} & +\frac{1}{c} \end{bmatrix} \begin{bmatrix} \tilde{I}_L \\ \tilde{v}_c \end{bmatrix}$

instability