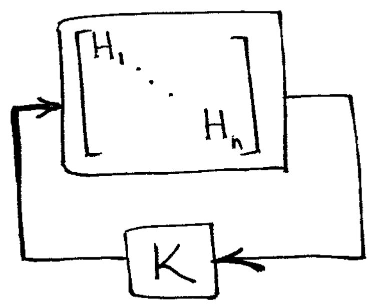
Nonlinear Systems

Lecture 24 04/25/13

from last time :



$$H_i: \quad \dot{x}_i = f(x_i) + g(x_i) \cdot u_i$$
$$y_i = h(x_i)$$
$$\dot{V}_i \leq -\varepsilon_i y_i^2 + y_i u_i$$

} scalar inputs and outputs

$K \in \mathbb{R}^{n \times n}$ interconnection matrix

H_i : uncoupled

K : provides coupling between them

$$V = \sum d_i V_i$$

$$P := \operatorname{diag}\{d_i\} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

$$A := -\operatorname{diag}\{\varepsilon_i\} + K = -\begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{bmatrix} + K$$

Stability of fbK interconnection (sufficient condition) amounts to existence of diagonal P s.t.

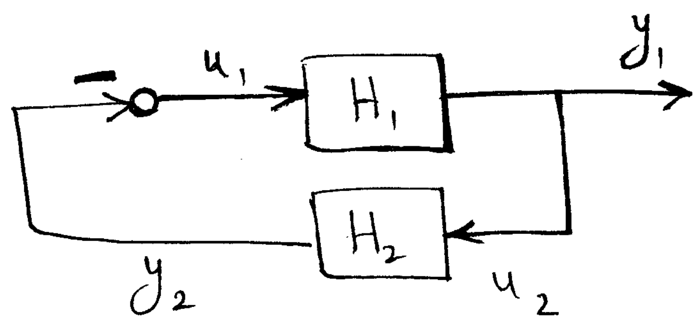
$$A^T P + PA < 0$$

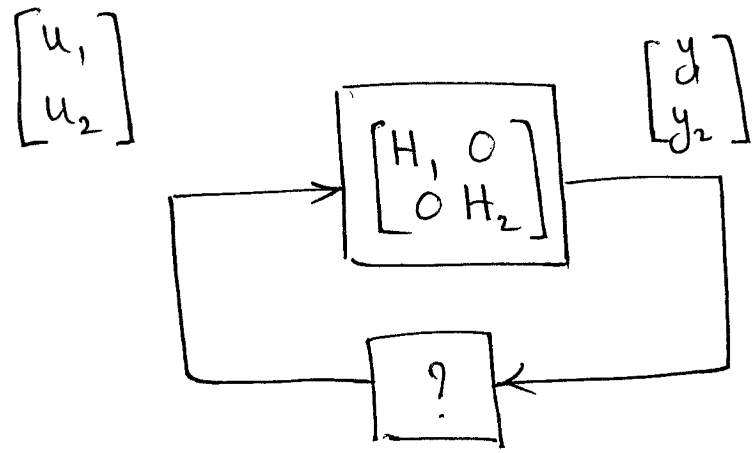
$$-\begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{bmatrix} + K$$

$$\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

use cvx (Stephen Boyd...) to solve this feasibility problem.

Ex 2





$u_1 = -y_2$

$u_2 = y_1$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} & ? & \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

things we know:

$H_i \rightarrow \dot{V}_i \leq -\varepsilon_i y_i^2 + y_i u_i$

Note! $\quad K + K^T = 0 \qquad$ (K is a skew hermitian matrix)

Need to check existence of a diagonal P s.t.

$$\left(-\begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix} + K^T\right)P + P\left(-\begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix} + K\right) < 0 \quad \ldots (I)$$

last time we showed that :

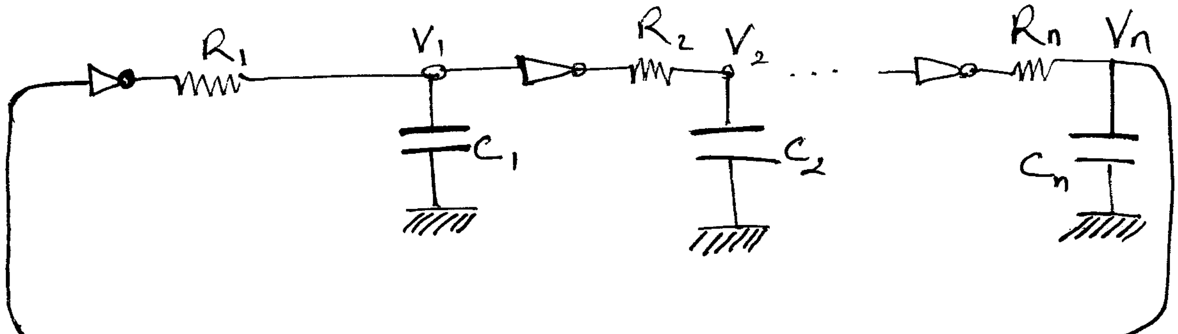
$V = V_1 + V_2$ works!

$$\Rightarrow \quad P = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \dots \quad (\text{II})$$

$$(\text{II}) \rightarrow (\text{I}) \Rightarrow -\begin{bmatrix} 2\varepsilon_1 & 0 \\ 0 & -2\varepsilon_2 \end{bmatrix} \underbrace{\cancel{+K^T+K}}_{0}$$

→ If H_i's are nonlinearities (static) you would be able to find linear combinations of V's as storage function provided that they are passive...

Ex n-stage ring oscillator



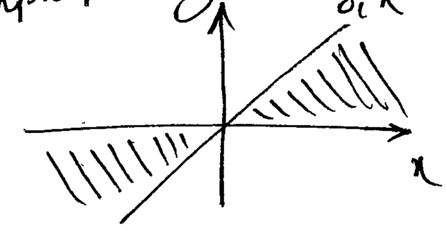
nonlinearities

$$
\left.
\begin{aligned}
R_1 C_1 \dot{V}_1 &= -V_1 - h_n(V_n) \\
R_2 C_2 \dot{V}_2 &= -V_2 - h_1(V_1) \\
&\vdots \\
R_n C_n \dot{V}_n &= -V_n - h_{n-1}(V_{n-1})
\end{aligned}
\right\}
$$

1) $x h_i(x) > 0$

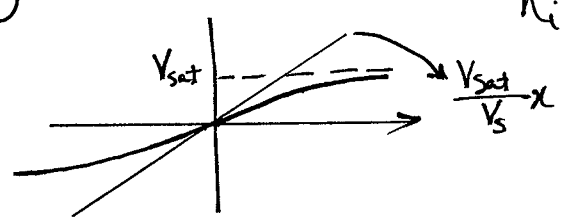
2) $x h_i(x) \leq \delta_i x^2$

input passivity



Typically :

$$h_i(x) = V_{sat} \cdot \tanh\left(\frac{x}{V_s}\right)$$



choose storage functions as :
$$V_i(x_i) = R_i C_i \int_0^{x_i} h_i(\xi)\, d\xi$$

$$H_i : \quad R_i C_i \dot{x}_i = -x_i + u_i$$
$$y_i = h_i(x_i)$$

where $u_i = -y_{i-1}\Big|_{\text{mod } n}$ gives the coupling in the model

$$K = \begin{bmatrix} 0 & 0 & \cdots & & -1 \\ -1 & 0 & & & \\ & -1 & 0 & & \\ & 0 & & \ddots & \\ & & & -1 & 0 \end{bmatrix}$$

$$\boxed{\dot{V}_i} = R_i C_i h_i(x_i)\dot{x}_i = -x_i h_i(x_i) + \overset{y_i}{h_i(x_i)}\, u_i$$

$$\boxed{= -x_i h_i(x_i) + y_i u_i}$$

$$x h_i(x) \leq \delta_i \cdot x^2$$

$$x(h_i(x) - \delta_i x) \leq 0 \quad \Rightarrow \quad \begin{array}{l} x > 0, \ h_i(x) \leq \delta_i x \\ x < 0, \ h_i(x) \geq \delta_i x \end{array}$$

$$x h_i(x) \quad \Rightarrow \quad \begin{array}{ll} x > 0 & h_i^2(x) \leq \delta_i x h_i(x) \\ x < 0 & h_i^2(x) \leq \delta_i x h_i(x) \end{array}$$

so we can always lower bound $\delta_i x h_i(x)$

$$\Rightarrow \quad x h_i(x) \geq \frac{1}{\delta_i} h_i^2(x)$$

$$-x h_i(x) \leq -\frac{1}{\delta_i} \underbrace{h_i^2(x)}_{y_i^2}$$

output strictly passive!

Therefore

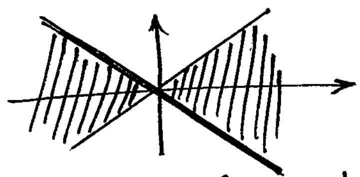
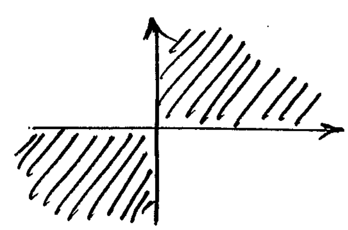
$$\dot{V}_i \leq -\frac{1}{\delta_i} y_i^2 + y_i u_i \qquad (*)$$

So stability of this interconnection is satisfied if we can find a matrix P that satisfies the lyapunov equation with the structured K matrix provided and that $(*)$ is satisfied.

$$V_i\text{'s came from } -R_iC_i\int_0^{x_i} h_i(s)\,ds$$

where R_iC_i is the time constant of each individual subsystem.

Summary

	L_2 gain	Passivity
Definition	$\exists \gamma, \beta$ st. $\|y_T\|_2 \le \gamma \|u_T\|_2 + \beta$	$\langle u_T, y_T \rangle \ge -\beta$ forall T
state space verification with storage fcn $V(x)$	$\dot V \le -\frac{1}{2} y^T y + \frac{\gamma}{2} u^T u$	$\dot V \le u^T y$ (or $y^T u$)
Equivalent condition for $\dot x = f(x) + g(x) u$, $y = h(x)$	$\frac{\partial V}{\partial x} f(x) + \frac{1}{2} h^T(x) h(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g(x) g^T(x) \frac{\partial V}{\partial x} \le 0$ (HJ)	$\frac{\partial V}{\partial x} f(x) \le 0$ $\frac{\partial V}{\partial x} g(x) = h^T(x)$
linear case $\dot x = Ax + Bu$, $y = Cx$	$A^T P + PA + C^T C + \frac{1}{\gamma^2} PBB^T P \le 0$ (Bounded real lemma)	$A^T P + PA \le 0$ $PB = C^T$ (KYP lemma)
Freq. domain condition $H(s) = C(sI - A)^{-1} B$	$\|H(j\omega)\| < \gamma$ forall ω	$\mathrm{Re}\{H(j\omega)\} \ge 0$ (Nyquist) ... (Bode) → phase properties
Memoryless	 Bow tie!	
Stability thm for interconn.	Small gain thm.	Passivity thm.