Nonlinear Systems

Lecture 06 02/07/13

Last time:

- Bendixon thm (examples)
- Invariant Sets (definition)
- Poincare-Bendixon Thm

Today:

- Hopf bifurcations
  - super ☺
  - sub ☹
- Nondimensionalization
- Center manifold theory (if time permits)

Back to bifurcations

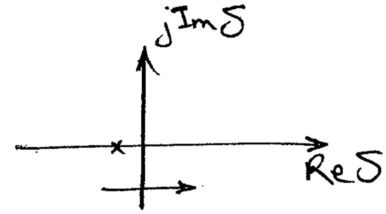
So far, we've covered 3 types:

- Fold $\dot{x} = \alpha \pm x^2$
- Transcritical $\dot{x} = \alpha x \pm x^2$
- Pitchfork $\dot{x} = \alpha x \pm x^3$
  - super
  - sub

Even though all of them can appear in higher dimensions (systems with multiple states $(n>1)$) they are essentially 1D.
(1D subspace captures behavior)

In all 3 cases, linearization @ the critical value of α vanishes:

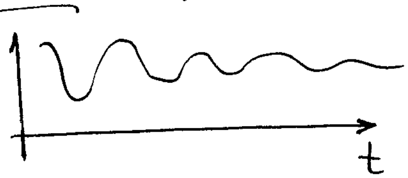
$$A = \left.\frac{\partial f}{\partial x}\right|_{\alpha_c, \bar{x}(\alpha_c)} = 0$$



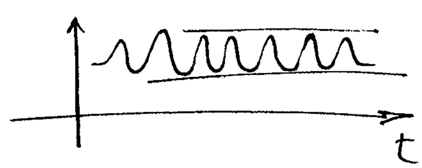
Supercritical Hopf bifurcation involves loss of stability of an e.p. which is stable focus and formulation of a stable limit cycle.

time dependence

$\alpha < \alpha_c$

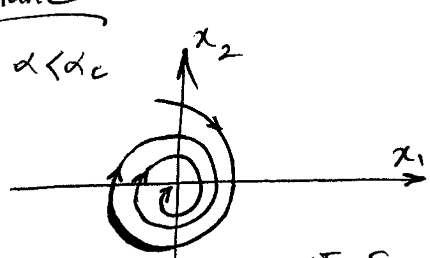


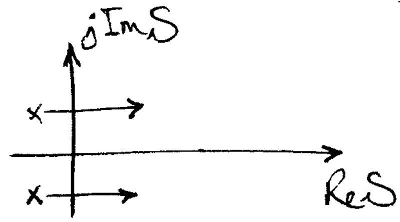
$\alpha > \alpha_c$



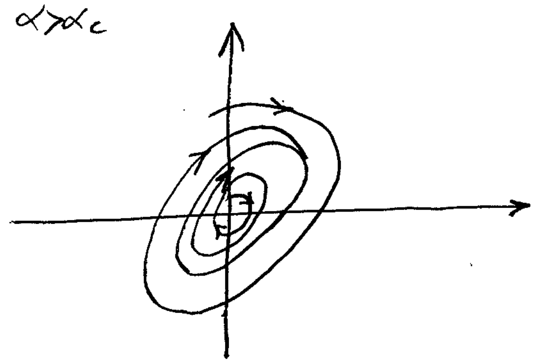
phase plane

$\alpha < \alpha_c$





$\alpha > \alpha_c$



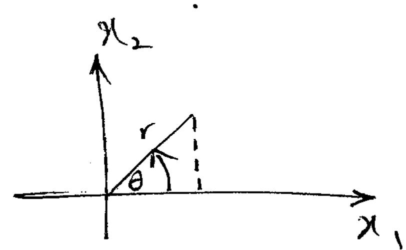
Ex.

$$\dot{x}_1 = x_1(\alpha - x_1^2 - x_2^2) - x_2$$
$$\dot{x}_2 = x_2(\alpha - x_1^2 - x_2^2) + x_1$$

Polar coordinates:

$x_1 = r\cos\theta$

$x_2 = r\sin\theta$



$\dot{r} = \alpha r - r^3$

$\dot{\theta} = 1$

$\dot{r} = 0 \Rightarrow \bar{r}(\alpha - \bar{r}^2) = 0$

$\bar{r} = 0$ or $\bar{r} = \sqrt{\alpha}$

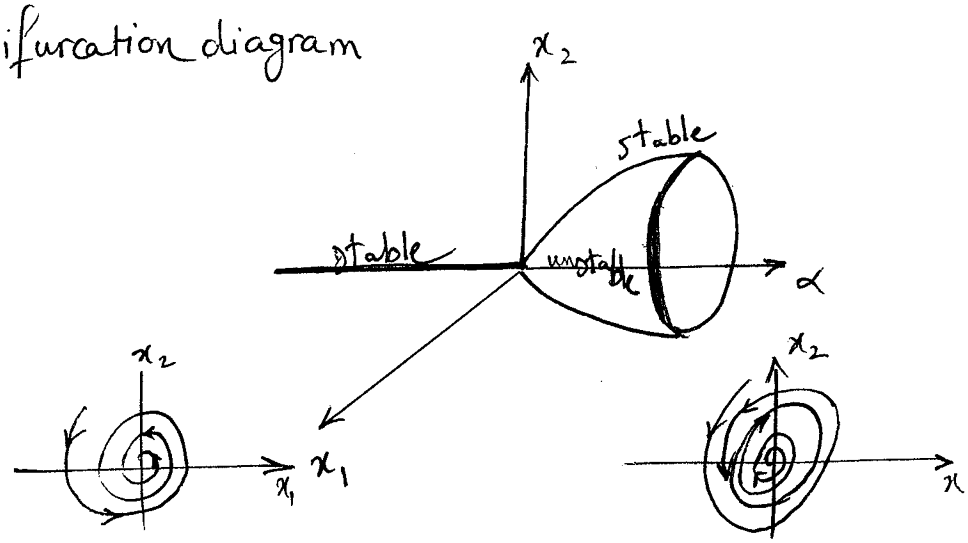
$\alpha > 0$

Summary:

$\alpha < 0 \Rightarrow \bar{r} = 0$ (unique e.p.)

$\alpha > 0 \Rightarrow \bar{r} = 0$ (e.p.) ; $\bar{r} = \sqrt{\alpha}$ limit cycle

Bifurcation diagram



hopf bifurcation

Q: Why is supercritical not a dramatic change?

A: even though we lost stability of the origin (for $\alpha > \alpha_c = 0$) if α is small positive number then departure from the origin will be small aswell (radius of limit cycle $\sqrt{\alpha}$)

Subcritical Hopf

$$\dot{r} = \alpha r + r^3 - r^5$$
$$\dot{\theta} = 1$$

$$\left[\begin{array}{l} \text{Khalil for} \\ \dot{x}_1 = \cdots \\ \dot{x}_2 = \cdots \end{array}\right]$$

Aside

linearization was

$$A = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix}$$

$$\lambda_{1,2} = \alpha \pm j$$

$\dot{r} = 0 \Rightarrow$

$$\bar{r}\,(\alpha + \bar{r}^2 - \bar{r}^4) = 0$$

→ if $\alpha < 0$ linearization around origin is stable (locally asymp. stable) because αr dominates the effect of $r^3 - r^5$ whe r is small.

→ if $\alpha > 0$ origin will be unstable e.p.

$$\dot{r} = 0 \qquad \bar{r}\,(\alpha + \bar{r}^2 - \bar{r}^4) = 0$$

Solutions

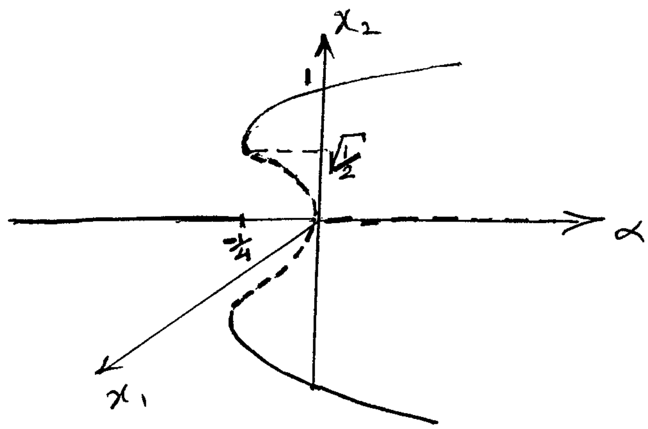
$\bar{r} = 0$ (e.p.)

$$\bar{q}^2 - \bar{q} + \alpha = 0 \quad , \quad \bar{q} = \bar{r}^2$$

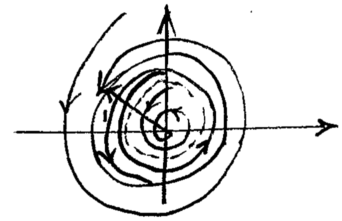
$$\bar{q}_{1,2} = \frac{1 \pm \sqrt{1 + 4\alpha}}{2} \Rightarrow \quad 1 + 4\alpha > 0 \qquad \alpha > -1/4$$

For $\alpha > -\frac{1}{4}$ additional "fixed points". (in r-equation) appear.

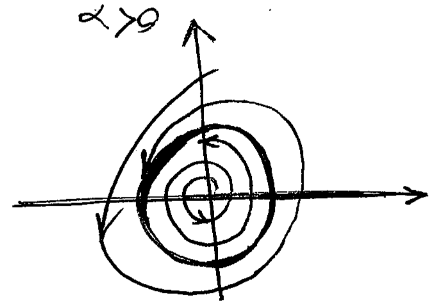
Bifurcation diagram:



$-\frac{1}{4} < \alpha < 0$



$\alpha > 0$



Big departure from origin for $\alpha > 0$!!! (nasty!)

departure from origin is of order "1" rather than $\sqrt{\alpha}$.

Non dimensionalization

$$\dot{x}_1 = -\alpha x_1 + \beta x_2$$

$$\dot{x}_2 = \frac{\gamma x_1}{\delta + x_1^2} - \eta x_2$$

Greek letters : parameters

Objective : (introduce scaling)

Scale x_1, x_2 & time (t) in order to reduce # of parameters.

$$z_1 = \frac{x_1}{X_1}$$

$$z_2 = \frac{x_2}{X_2}$$

$$\tau = \frac{t}{T}$$

X_1, X_2, T to be determined

$$\frac{\partial x_1}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial x_1}{\partial \tau} = \frac{1}{T}\frac{\partial x_1}{\partial \tau}$$

$$= \frac{X_1}{T}\frac{\partial z_1}{\partial \tau}$$

$$\Rightarrow \quad \frac{dz_1}{d\tau} = \frac{T}{X_1}\left[-\alpha X_1 z_1 + \beta X_2 z_2\right]$$

$$\frac{dz_2}{d\tau} = \frac{T}{X_2}\left[\frac{\gamma X_1 z_1}{\delta + X_1^2 z_1^2}\right] - \frac{T}{X_2}\eta X_2 z_2$$

We can bring it to the following form

$$\frac{dz_1}{d\tau} = -a z_1 + z_2$$

$$\frac{dz_2}{d\tau} = \frac{z_1}{1+z_1^2} - b z_2$$

by proper choice of X_1, X_2, T.