Lecture 22   11/22/13

Last time:

- Reachability gramian

$$P_K = R_K \cdot R_K^T = \sum_{i=1}^{K-1} A^i B B^T \underbrace{(A^i)^T}_{(A^T)^i}$$

$$P_{K+1} = A P_K A^T + B B^T$$

For reachable systems

$$\text{minimize} \quad \underline{u}_K^T \underline{u}_K$$
$$\text{s.t.} \quad x_d - R_K \cdot \underline{u}_K = 0$$
$$k > n$$

$$x_d \leftarrow \boxed{R_K} \xleftarrow{\underline{u}_K} \boxed{R_K^T (R_K R_K^T)^{-1}} \leftarrow x_d$$

Today:

- Canonical form for unreachable systems (standard)
- Modal conditions for reachability (PBH test)

Recall: $x(k+1) = A\,x(k) + B \cdot u(k)$

reachable $\Leftrightarrow$ $\text{rank}(R_n) = n$

$$R_n = [\,A^{n-1}B \,\vdots\, \cdots \,\vdots\, B\,]$$

$$AR_n = [\,A^n R \,\vdots\, \cdots \,\vdots\, AB\,]$$

Cayley-Hamilton

$$A^n = -(a_{n-1}A^{n-1} + \cdots + a_0 I)$$

• Reachable subspace Range(R_n) = column span R_n

If $z \in$ Range(R_n) $\Rightarrow$ $A \cdot z \in$ Range(R_n)

Range(R_n) is A-invariant

Given a system

$$x(k+1) = A\,x(k) + B\,u(k) \quad \text{---- (1)}$$

with rank(R_n) $= r < n$ (unreachable)

Introduce a change of coordinates

$$x(k) = T \cdot z(k) \quad \text{---- (2)}$$

$$T \in \mathbb{R}^{n \times n}; \quad \det(T) \neq 0$$

$$T^{-1} \,|\; T \cdot z(k) = A \cdot T \cdot z(k) + B\,u(k)$$

$$z(k+1) = \bar{A} \cdot z(k) + \bar{B} \cdot u(k)$$

$$\boxed{\begin{array}{l} \bar{A} = T^{-1} A \cdot T \\ \bar{B} = T^{-1} B \end{array}} \iff \boxed{\begin{array}{l} T \cdot \bar{A} = A \cdot T \\ T \cdot \bar{B} = B \end{array}}$$

separate $T = [\underbrace{T_1}_{n \times r} \,\vdots\, \underbrace{T_2}_{n \times (n-r)}]$

Choose columns of T_1 to span reachable subspace (i.e. range(R_n))

$$\begin{aligned} x_d &= R_n \cdot \underline{u}_n \\ &= U \cdot \Sigma V^* \cdot \underline{u}_n \\ &= \sum_{i=1}^{r} \sigma_i u_i \cdot \underbrace{v_i^* \cdot \underline{u}_n}_{a_i} \end{aligned}$$

Range(R_n) = span$\{u_1, \cdots, u_r\}$

Thus, can select

$$T_1 = [\, u_1 \cdots u_r \,]$$

$$T_2 = [\, u_{r+1} \cdots u_n \,]$$

Partition $\bar{A}$ in the following way

$$T\bar{A} = AT$$

$$[\,T_1 \;\; T_2\,]\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = A\,[\,T_1 \;\; T_2\,]$$

$$[\, \underbrace{T_1 \cdot \bar{A}_{11} + T_2 \cdot \underbrace{\bar{A}_{21}}_{0}}_{= AT_1} \;\vdots\; \underbrace{T_1 \bar{A}_{12} + T_2 \cdot \bar{A}_{22}}_{= AT_2} \,] = [\, AT_1 \;\vdots\; AT_2 \,]$$

$\bar{A}_{21} = 0$, otherwise, we would get out of a reachable subspace.

$$[\,T_1 \;\; T_2\,]\begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} = T_1 \bar{B}_1 + T_2 \underbrace{\bar{B}_2}_{0} = B$$

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}\begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u(k)$$

$$z_2(k+1) = \bar{A}_{22}\, z_2(k)$$

$$\Downarrow$$

$$\boxed{z_2(k) = \bar{A}_{22}^{\,k} \cdot z_2(0)}$$ Can't touch with control!

$z_2(k) \in \mathbb{R}^{n-r}$ identifies unreachable components of the state vector $z(k)$

Thm: $x(k+1) = A\,x(k) + B\,u(k)$ unreachable

$$\Updownarrow$$

There is a left eigenvector of A $(w^T A = \lambda w^T)$
s.t. $w^T B = 0$

Proof: $\boxed{\text{“}\Uparrow\text{”}}$:

assume: there is w s.t. $\begin{cases} w^T A = \lambda w^T \\ w^T B = 0 \end{cases}$

$$w^T A B = \lambda \underline{w^T B} = 0$$

$$w^T A^2 B = \cdots = 0$$

$$\vdots$$

$$w^T A^{n-1} B = \cdots = 0$$

$$\Rightarrow w^T [\, A^{n-1} B \mid \cdots \mid B \,] = 0$$

$$\Rightarrow w^T \cdot R_n = 0$$

$\Rightarrow$ R_n is not a full rank matrix

rank $(R_n) < n$

$\Rightarrow$ Not reachable

$\boxed{\text{“}\Downarrow\text{”}}$:

assume: system is not reachable and show that

$\exists\, w$ s.t. $w^T A = \lambda w^T$
$w^T B = 0$

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} u(k)$$

Note: If $w^T \bar{A}_{22} = \lambda w^T$

Then $[\,0 \quad w^T\,] \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} = \lambda [\,0 \quad w^T\,]$

$$[\, 0\,\bar{A}_{11} + w^T \cdot 0 \mid 0\,\bar{A}_{12} + w^T \bar{A}_{22} \,] = [\, 0 \mid \lambda w^T \,]$$

$\oplus$ $[\,0 \quad w^T\,] \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} = 0$

PBH Test (Modal condition for reachability)

P → Popov
B → Belewitch
H → Hantus

$$x(k+1) = A\,x(k) + B\,u(k) \quad \text{reachable}$$

$$\Updownarrow$$

$$\text{rank}\left(\left[\, zI - A \;\vdots\; B \,\right]\right) = n \qquad \text{for all } z \in \mathbb{C}$$

Note: If z is not an eigenvalue of A, i.e. if

$\det(zI - A) \neq 0$

$\Rightarrow \text{rank}(zI - A) = n$

$\Rightarrow$ Only need to check for $z = \{\lambda_1(A), \cdots, \lambda_n(A)\}$

Ex: $\lambda_1 = 1$; $\lambda_2 = 2$, for $\bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$[\lambda_1 I - \bar{A} \;\vdots\; \bar{B}]$

$= \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right] \Rightarrow$ $\lambda_1 = 1$ reachable

$[\lambda_2 I - \bar{A} \;\vdots\; \bar{B}]$

$= \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$ $\lambda_2 = 2$ unreachable