Input-output norms of LTI systems

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Amplification of disturbances

- **Harmonic forcing**

$$d(t) \;=\; \hat{d}(\omega)\, \mathrm{e}^{\mathrm{j}\omega t} \;\xrightarrow{\text{steady-state response}}\; y(t) \;=\; \hat{y}(\omega)\, \mathrm{e}^{\mathrm{j}\omega t}$$

⋆ **Frequency response**

$$\hat{y}(\omega) \;=\; \underbrace{C\,(\mathrm{j}\omega I \;-\; A)^{-1}\,B}_{H(\omega)}\;\hat{d}(\omega)$$

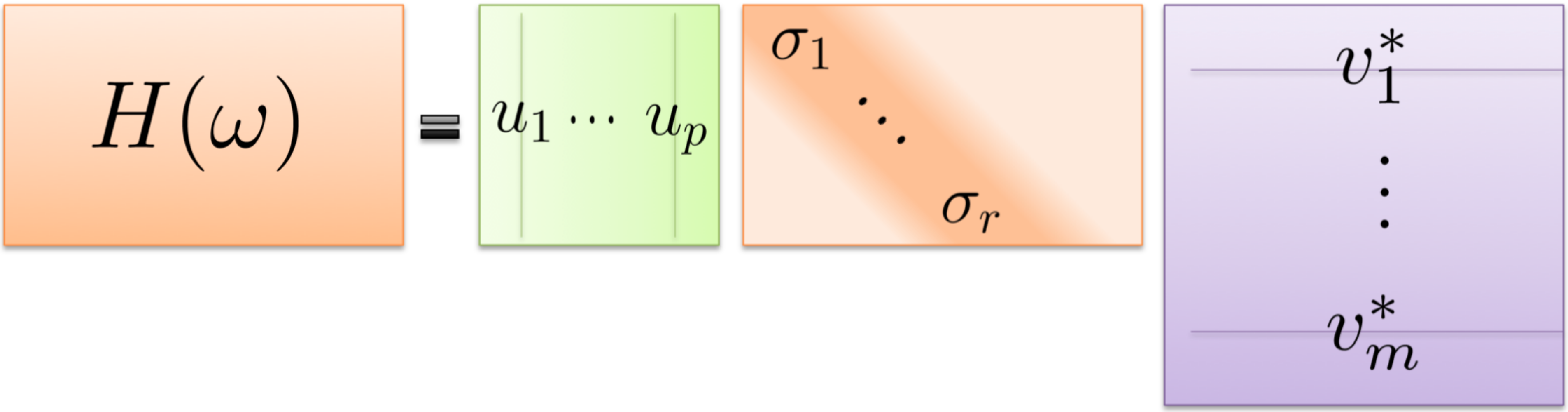
example: 3 **inputs,** 2 **outputs**

$$\begin{bmatrix} \hat{y}_1(\omega) \\ \hat{y}_2(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) & H_{13}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) & H_{23}(\omega) \end{bmatrix} \begin{bmatrix} \hat{d}_1(\omega) \\ \hat{d}_2(\omega) \\ \hat{d}_3(\omega) \end{bmatrix}$$

$H_{ij}(\omega)$ – response from jth input to ith output

Input-output gains

- **Determined by singular values of** $H(\omega)$



left and right singular vectors:

$$
\begin{aligned}
H(\omega)H^*(\omega)\, u_i(\omega) &= \sigma_i^2(\omega)\, u_i(\omega) \\
H^*(\omega)H(\omega)\, v_i(\omega) &= \sigma_i^2(\omega)\, v_i(\omega)
\end{aligned}
$$

$\{u_i\}$ orthonormal basis of output space

$\{v_i\}$ orthonormal basis of input space

- **Action of $H(\omega)$ on $\hat{d}(\omega)$**

$$\hat{y}(\omega) \;=\; H(\omega)\,\hat{d}(\omega) \;=\; \sum_{i=1}^{r} \sigma_i(\omega)\, u_i(\omega) \left\langle v_i(\omega), \hat{d}(\omega) \right\rangle$$

- **Right singular vectors**
  - ⋆ **identify input directions with simple responses**

$$\sigma_1(\omega) \;\geq\; \sigma_2(\omega) \;\geq\; \cdots \;>\; 0$$

$$\hat{y}(\omega) \;=\; \sum_{i=1}^{r} \sigma_i(\omega)\, u_i(\omega) \left\langle v_i(\omega), \hat{d}(\omega) \right\rangle$$

$$\Big\downarrow \; \hat{d}(\omega) \;=\; v_k(\omega)$$

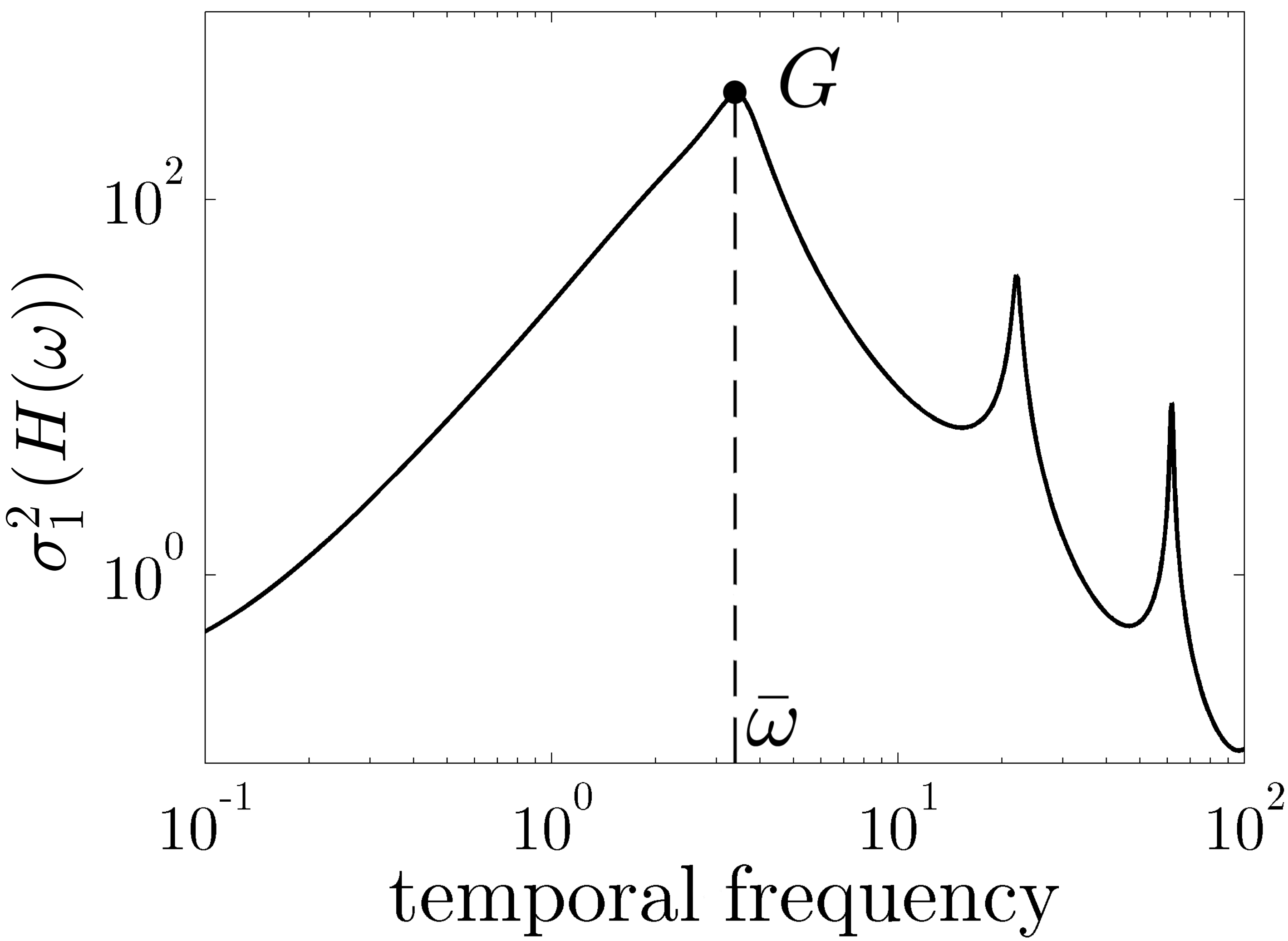
$$\hat{y}(\omega) \;=\; \sigma_k(\omega)\, u_k(\omega)$$

$\sigma_1(\omega)$**: the largest amplification at any frequency**

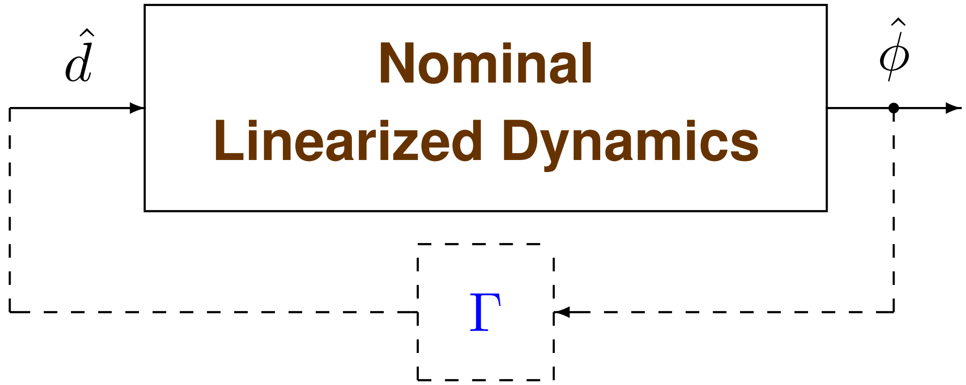
Worst case amplification

- H_∞ **norm: an induced L_2 gain (of a system)**

$$G \;=\; \|H\|_\infty^2 \;=\; \max \frac{\text{output energy}}{\text{input energy}} \;=\; \max_{\omega} \sigma_1^2\left(H(\omega)\right)$$



Robustness interpretation: small-gain theorem



modeling uncertainty

(can be nonlinear or time-varying)

- **Closely related to pseudospectra of linear operators**

$$\dot{x}(t) \;=\; (A \,+\, B\,\Gamma\,C)\,x(t)$$

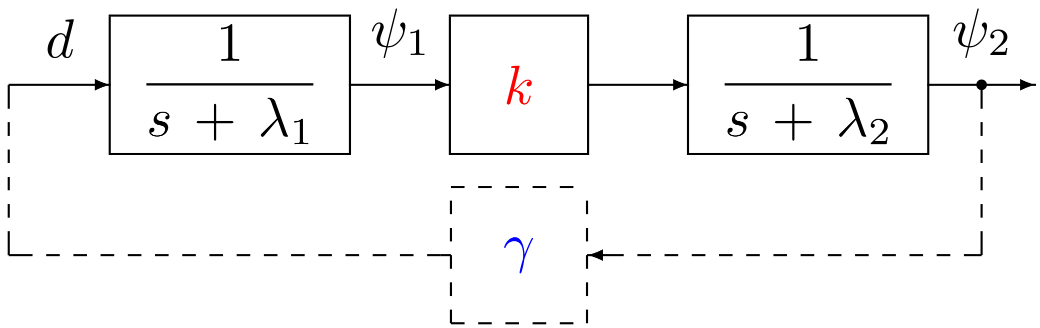
LARGE **worst case amplification** $\Rightarrow$ **small** **stability margins**

Back to a toy example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & \mathbf{0} \\ k & -\lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d$$

$$G = \max_{\omega} |H(\mathrm{j}\omega)|^2 = \frac{k^2}{(\lambda_1 \lambda_2)^2}$$

ROBUSTNESS



modeling uncertainty

stability

$\Updownarrow$

$\gamma < \lambda_1 \lambda_2 / k$

$$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\lambda_1 & \gamma \\ k & -\lambda_2 \end{bmatrix} \right) = s^2 + (\lambda_1 + \lambda_2)\, s + \underbrace{(\lambda_1 \lambda_2 - \gamma\, k)}_{> 0}$$

Response to stochastic forcing

- **White-in-time forcing**

$$\mathcal{E}\left(d(t_1)\, d^*(t_2)\right) \;=\; I\, \delta(t_1 \,-\, t_2)$$

⋆ **Frobenius norm**

power spectral density:

$$\|H(\omega)\|_F^2 \;=\; \mathrm{trace}\left(H(\omega)\, H^*(\omega)\right) \;=\; \sum_{i\,=\,1}^{r} \sigma_i^2\,(\omega)$$

⋆ H_2 **norm**

variance amplification:

$$\|H\|_2^2 \;=\; \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \|H(\omega)\|_F^2\, \mathrm{d}\omega \;=\; \int_0^{\infty} \|H(t)\|_F^2\, \mathrm{d}t$$

Computation of H_2 and H_∞ norms

$$
\begin{aligned}
\dot{x}(t) &= A\,x(t) + B\,d(t) \\
y(t) &= C\,x(t)
\end{aligned}
$$

- **H_2 norm**

  ⋆ **Lyapunov equation**

$$
\mathcal{E}\left(d(t_1)\,d^*(t_2)\right) = W\,\delta(t_1 - t_2) \;\Rightarrow\; \left\{ \begin{aligned} \|H\|_2^2 &= \operatorname{trace}\left(C\,P\,C^*\right) \\ A\,P + P\,A^* &= -\,B\,W\,B^* \end{aligned} \right.
$$

- **H_∞ norm**

  ⋆ **E-value decomposition of Hamiltonian in conjunction with bisection**

$$
\|H\|_\infty \geq \gamma \;\Leftrightarrow\; \begin{bmatrix} A & \frac{1}{\gamma}\,B\,B^* \\ -\frac{1}{\gamma}\,C^*\,C & -A^* \end{bmatrix}
$$
has at least one imaginary e-value