Lecture 187 11/5

today: $A^T P + PA = -Q$ (Lyapunov Eq)

$\forall\, Q = Q^T > 0 \;\; \exists\, P = P^T > 0 \iff \dot{x} = Ax$: stable

$P = \int_0^\infty e^{A^T t} Q e^{At}\, dt$

Lyapunov function $V(x) = x^T P x$

In HW #5, we'll do similiar things for LTI systems in discrete time : $V(k) = x^T(k) P x(k)$

So far → stability

coming up → performance

* system norms (input/output norms)

$u \longrightarrow \boxed{\text{LTI system}} \longrightarrow y$

In order to talk about performance, we need to decide how to "measure the size" of input + output signals

Signal norms: measure size of signal

Consider SISO system :

$u \longrightarrow \boxed{\text{SISO system (stable)}} \longrightarrow y$

$u(t) \in \mathbb{R}$

$y(t) \in \mathbb{R}$

Transfer Function

true in MIMO

$\Rightarrow y(t) = \int_0^t H(t-\tau)\, u(\tau)\, d\tau$

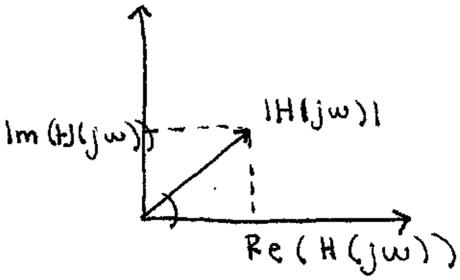
$Y(S) = H(S)\, U(S)$

Laplace Transform

In SISO case: $H(S) = Y(S)/U(S) \in \mathbb{C} \quad \forall\, S \in \mathbb{C}$

for $S = j\omega \Rightarrow H(j\omega) = Y(j\omega)/U(j\omega) \in \mathbb{C} \quad \forall\, \omega \in \mathbb{R}$

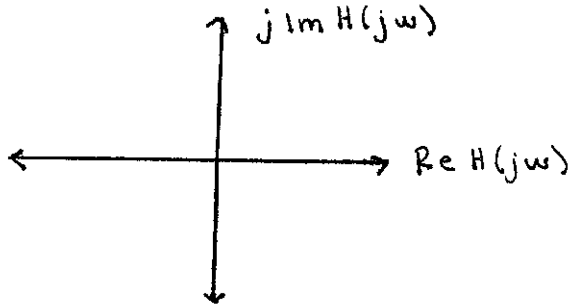
temporal freq

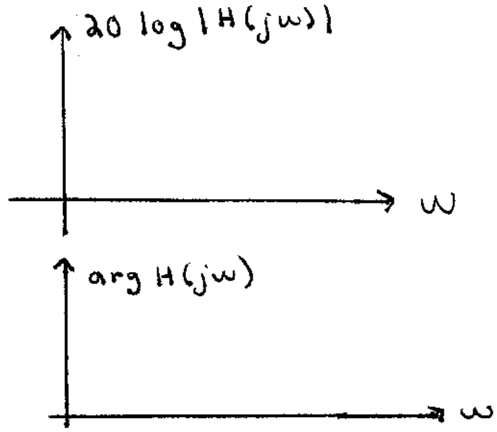


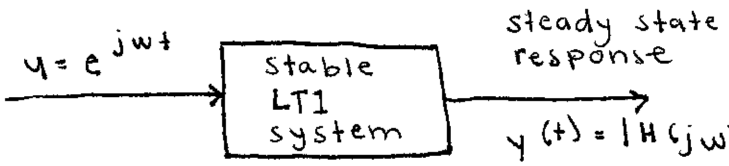
$\nearrow$ Nyquist

$$H(j\omega) = \mathrm{Re}\, H(j\omega) + j\, \mathrm{Im}(H(j\omega))$$
$$= |H(j\omega)| \cdot e^{j \arg H(j\omega)}$$

↳ Bode







$$y(t) = |H(j\omega)|\, e^{j(\omega t + \arg H(j\omega))}$$

$|H(j\omega)|$: amplifications at any frequency

$\arg H(j\omega)$: phase lag —‖—

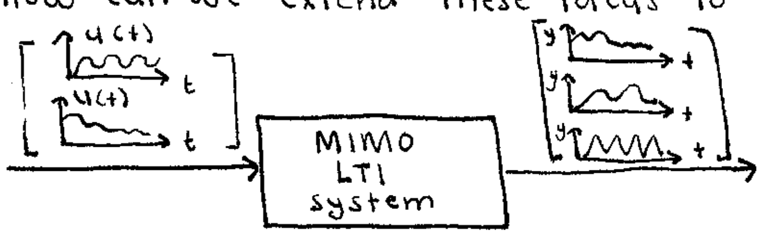
(there is also an appealing time-domain interpretation)

ex// $H(s) = \frac{K}{s+1} \Leftrightarrow \dot{y} + y = K \cdot u \Rightarrow H(j\omega) = \frac{K}{\sqrt{\omega^2+1}}$

$|H(j\omega)| = \left((\mathrm{Re}\, H(j\omega))^2 + (\mathrm{Im}(H(j\omega)))^2 \right)^{1/2}$ → low-pass filter

$K > 0 \Rightarrow \arg H(j\omega) = -\mathrm{atan}\, \omega$

Q: How can we extend these ideas to MIMO?

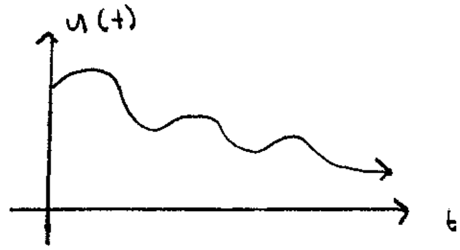


$$\rightarrow \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \\ H_{31} & H_{32} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

$$Y_i(s) = \sum_{j=1}^{m} H_{ij}(s) \cdot U_j(s) = H_{i1}(s) \cdot U_1(s) + H_{i2}(s)\, U_2(s)$$

How should we weigh input channels to get biggest output?

Signal Norms:



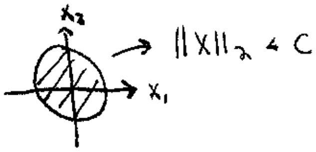
vector norms:

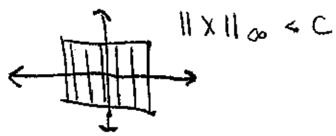
given $x \in \mathbb{R}^n$, how can you measure it's size?

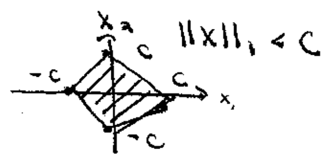
$\hookrightarrow \|X\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$

$\|x\|_\infty = \max |x_i|$

$\|x\|_1 = \sum_{i=1}^{n} |x_i| = |x_1| + \dots + |x_n|$







More generally, P-norm : $P \geq 1$

$\|x\|_p^p = |x_1|^p + \dots |x_n|^p$

At any fixed time, input + output signals are vectors.

Thus $\hookrightarrow$ $\|u(t)\|_2^2 = u_1^2(t) + \dots + u_m^2(t)$

$\|u(t)\|_1 = \sum_{i=1}^{m} |u_i(t)|$

Q: What do we do with time?

L_2-norm : energy

$\hookrightarrow$ $\underbrace{\|u\|_2^2}_{\text{signal norm}} = \int_0^\infty \overbrace{\|u(t)\|_2^2}^{\text{vector norm}} \, dt$ $= \int_0^\infty u^T(t)\, u(t)\, dt$

* key: u evaluated @ t is a vector $\mathbb{R}^m$

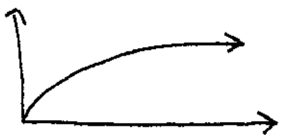
L_1-norm: action

$\|u\|_1 = \int_0^\infty \|u(t)\|_1 \, dt = \int_0^\infty \sum_{i=1}^{m} |u_i(t)| \, dt$

L_∞-norm:

$\|u\|_\infty = \sup_t \|u(t)\|_\infty$ → glorified maximum (least upper bound) $\hookrightarrow \frac{d}{dt} = 0$?

ex//



$1-e^{-t}$

smallest / least upper bound = 1

$$\frac{d}{dt}\left(1-e^{-t}\right) = e^{-t} \neq 0$$

Power of a Signal:

$$\|u\|_{power} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \|u(t)\|_2^2 \, dt$$

↳ not a norm