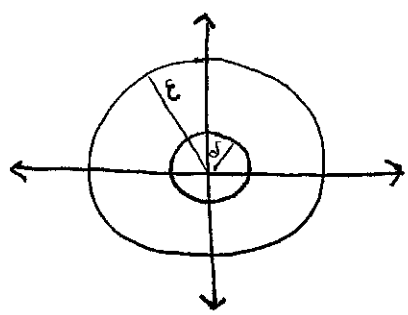
Lecture 14 10/22

let $\dot{x} = f(x)$ have an e.p. @ $\overline{x} = 0$.
then $\overline{x} = 0$ is (locally) asymptotically stable if:

1. for any $\varepsilon > 0$, there is $\delta_1 > 0$ such that: $\delta_1 < \varepsilon$
   $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon$ for all $t \geq 0$
   (stability of $\overline{x} = 0$)
2. there is $\delta_2 > 0$ such that for all
   $\|x(0)\| < \delta_2 \Rightarrow x(t) \xrightarrow{t \to \infty} 0$
   $(\lim_{t\to\infty} \|x(t)\| = 0)$

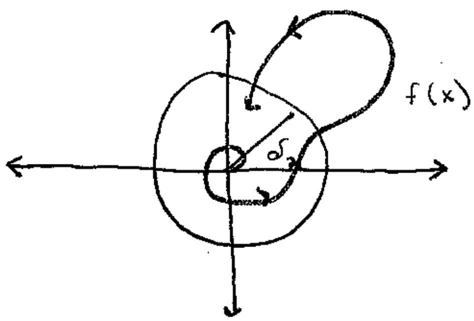
Illustration:

1.)



starting @ δ_1, don't exceed ε
"start close, stay close"

2.)



"attractiveness"

If 1 holds ✓ but 2 doesn't X $\Rightarrow \overline{x} = 0$ is stable
If 1 ✓ and 2 ✓ $\Rightarrow \overline{x} = 0$ is locally assympt. stable
If 1 ✓ and 2 ✓ ⊕ $\delta_2 = \pm\infty \Rightarrow \overline{x} = 0$ is globally assympt. stable
If 1 X $\Rightarrow \overline{x} = 0$ is unstable
If 1 X ; 2 ✓ $\Rightarrow \overline{x} = 0$ is attractive

note: If there are multiple equilibrium points, then neither of them can be globally assymptotically stable.

proof: start @ another equilibrium point

Examples!

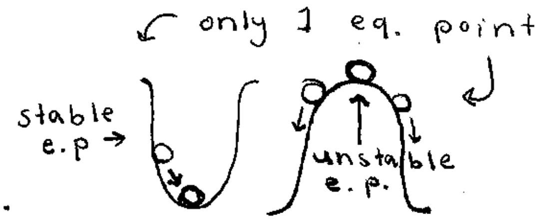
ex// $\dot{x} = x^2 - x \; : \; x(t) \in \mathbb{R}$

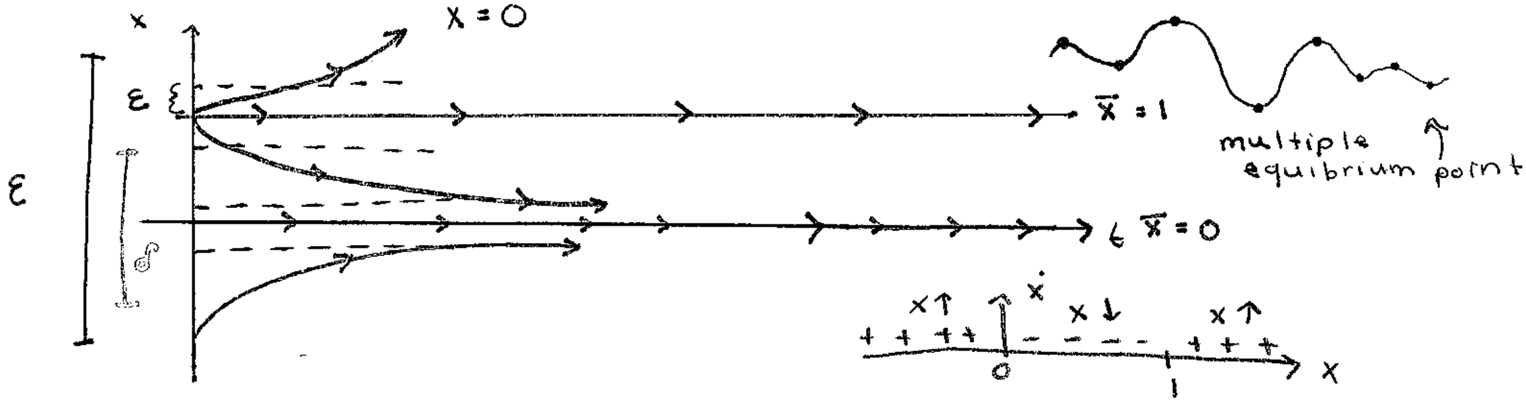
$f(x) = x(x-1)$

$f(\bar{x}) = 0 \Rightarrow x = 1$

$x = 0$

* not globally A.S.



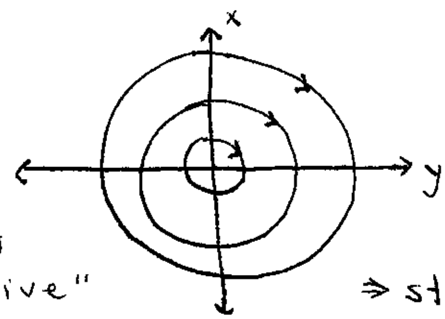


thus $\bar{x} = 0$: locally asymptotically stable

$\bar{x} = 1$: unstable

ex// $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

MS system



solutions: sins + cosines

1.) holds

2.) doesn't hold

can see system is no "attractive"

⇒ stable, not asymptotically

NOTE: if $\dot{x} = Ax$: stable

$\Rightarrow \bar{x} = 0$ globally asymptotically stable

$(\mathrm{Re}(\lambda_i(A)) < 0)$

if marginally stable (system)

$\Rightarrow \bar{x} = 0$ is stable but not attractive

if unstable

$\Rightarrow \bar{x} = 0$ is unstable

Whenever you don't have linear systems
→ LINEARIZE THEM

fact: Linearization can provide insight into local stability stability properties of equilibrium points on nonlinear systems.

$\dot{x} = f(x)$ with eq. point $\bar{x}$

↳ If $A = \frac{\partial F}{\partial x}\Big|_{x=\bar{x}} \Rightarrow \dot{\tilde{x}} = A\tilde{x}$

then ...

1.) If $\text{Re}(\lambda_i(A)) < 0 \Rightarrow \bar{x}$ of $\dot{x} = f(x)$ is locally assympt. stable
2.) If there is $\text{Re}(\lambda_i(A)) > 0 \Rightarrow \bar{x}$ is unstable
3.) If linearization is marginally stable ⇒ addition analysis is needed

ex// Back to $\dot{x} = x^2 - x$

$$\partial f/\partial x\big|_{\bar{x}} = 2x - 1\big|_{\bar{x}} = \begin{cases} -1, & \bar{x} = 0 \\ +1, & \bar{x} = 1 \end{cases}$$

$$\dot{\tilde{x}} = \begin{cases} -\tilde{x}, & \bar{x} = 0 \quad \text{(stable)} \\ +\tilde{x}, & \bar{x} = 1 \quad \text{(unstable)} \end{cases}$$

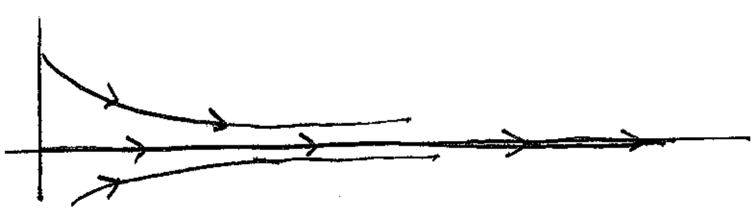
⇒ $\bar{x} = 0$: locally as. stable
$\bar{x} = 1$: unstable

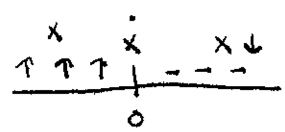
ex// a. $\dot{x} = -x^3$
b. $\dot{x} = +x^3$

$$\left.\begin{array}{l} \text{a. } \dot{x} = -x^3 \\ \text{b. } \dot{x} = +x^3 \end{array}\right\} \frac{\partial f}{\partial x}\Big|_{\bar{x}=0} = \mp 3\cdot\bar{x}^2\Big|_{\bar{x}=0} = 0$$

$\Rightarrow \dot{\tilde{x}} = 0\cdot\tilde{x}$

a.)





b.