Lecture 13 10/17

Last time: Modal Conditions for stability of LTI systems

Today:

thm: An LTI system: $\dot{x} = Ax$ is stable if + only if $Re(\lambda_i(A)) < 0$ for all $i = 1, \ldots, n$
(read: all trajectories are bounded for all t and they go to zero as $t \to \infty$)

- Marginally stable if $Re(\lambda_i(A)) \le 0$ for all $i = 1, \ldots n$ and the Jordan blocks correspond to the $Re(\lambda_i(A)) = 0$ are of size 1×1.
(read: all trajectories are bounded for all t)
- unstable if + only if there is i s.t. $Re(\lambda_i(A)) > 0$ or if there is i s.t. $Re(\lambda_i(A)) = 0$ the size of the corresponding Jordan block is ≥ 2
(read: there is a trajectory that goes to ∞ as $t \to \infty$)

↳ key observation: $e^{At} = T e^{Jt} T^{-1}$

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_m t} \end{bmatrix}$$

$$e^{\lambda_i t} z_i(0) = e^{(Re \lambda_i) t} e^{j (Im \lambda_i) t} z_i(0)$$

In DT time, replace $Re(\lambda_i(A))$ w/ $|\lambda_i(A)|$
replace < 0 with < 1

In classical control course: "Routh-Hurwitz criterion" for stability of CT LTI system

↳ uses $\det(sI - A) = s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$ (must be 1, if not, normalize)
coefficients in the characteristic polynomial to establish conditions for stability

** $a_n = 1$!!!

ex// $n = 2$: $f(s) = s^2 + a_1 s + a_0$

$\left.\begin{array}{l} a_1 > 0 \\ a_0 > 0 \end{array}\right\} \Rightarrow$ necessary for stability

Particular case : $A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \Rightarrow \begin{bmatrix} s & -1 \\ a_0 & s+a_1 \end{bmatrix}$

ex// $n = 3$: $f(s) = s^3 + a_2 s^2 + a_1 s + a_0$

$a_i > 0$, $i = 1, 2, 3$

$a_2 \cdot a_1 > a_0 \cdot 1$

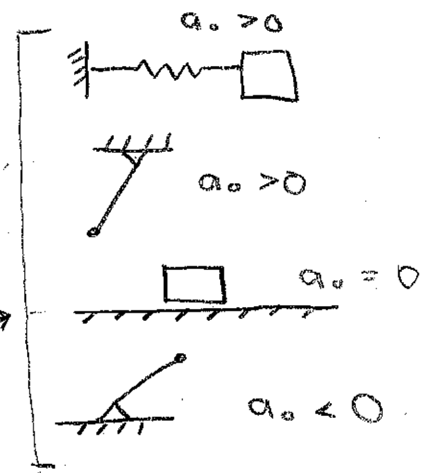
ex// $f(s) = s^2 + a_1 s + a_0$

a.) for stability : $a_1 > 0$ (RH)

$a_2 > 0$

b.) $a_1 = 0 \rightarrow A = \begin{bmatrix} 0 & 1 \\ -a_0 & 0 \end{bmatrix}$

$f(s) = s^2 + a_0 \Rightarrow$



$\Rightarrow$ marginal stability $a_0 > 0$

unstable $a_0 \leq 0$

c.) $a_0 = 0$

$\Rightarrow f(s) = s^2 + a_1 s = s(s + a_1) = 0$

$\Rightarrow s_1 = 0$

$s_2 = -a_1$

$\Rightarrow a_1 > 0$ marginal stability

$a_1 < 0$ unstable

Stability of equilibrium points of non-linear system

$\dot{x} = f(x)$

eq. point : constant trajectory

$x(t) = \bar{x} = \text{const} \Rightarrow$

$\dot{\bar{x}} = d\bar{x}/dt = 0$

$f(\bar{x}) = 0$

In linear case : $A\bar{x} = 0$

in linear case: $A\bar{x} = 0$

$\Rightarrow$ if $\det(A) \neq 0$ then $\bar{x} = 0$ is a unique eq. point

$\Rightarrow$ if $\det(A) = 0$ then there are ∞-many eq. points

↳ ($\exists$ e-value of A @ origin)

↳ eq. point determined by $\text{Null}(A) := \{x \text{ s.t. } A \cdot x = 0\}$

ex// Mass-spring system

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix}$$

$\det(A) \neq 0 = k/m$ (for $k \neq 0$)

$\Rightarrow$ unique eq. point $\bar{x} = 0$

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $k = 0 \Rightarrow A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\det(A) = 0$

$\Rightarrow$ infinitely many eq. points

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \bar{x}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \bar{x}_2 = 0 \\ \bar{x}_1 \in \mathbb{R} \end{matrix}$$

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}$$ → any real number

For non-linear case → things more "exotic"

a.) one eq. point

$\dot{x} \Rightarrow x^3 \Rightarrow x^3 = 0$

b.) finite number of eq. points

$\dot{x} = x(x-1) \Rightarrow \bar{x} = 0$ or $\bar{x} = 1$

c.) infinitely many eq. points

$$\begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2^3 \end{matrix} \Rightarrow \bar{x} = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

d.) no eq. points

$\dot{x} = x^2 + 1$

If there is an eq. point at $\bar{x} \neq 0$, we can introduce a change of coordinates to bring it to the origin.

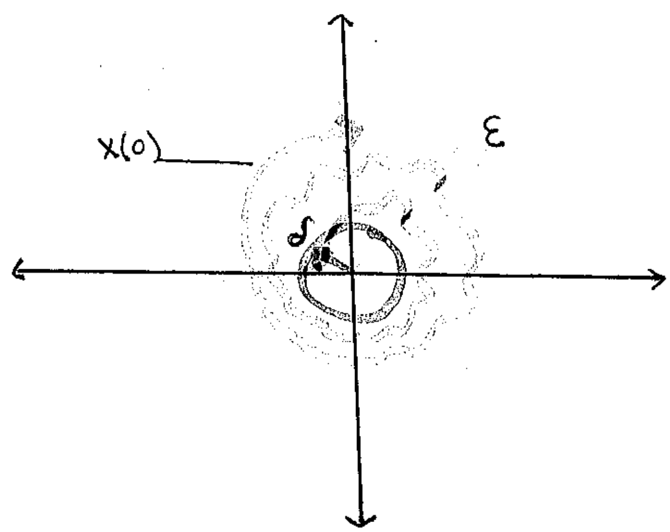
$$\left.\begin{array}{l} \dot{x} = f(x) \\ z = x - \bar{x} \end{array}\right\} \Rightarrow \dot{z} = \dot{x} - \dot{\bar{x}} = f(x) = f(z + \bar{x})$$

Since $f(\bar{x}) = 0 \Rightarrow \bar{z} = 0$ is equil. point

Definition of Stability: (of eq. point)

Let $\dot{x} = f(x)$ have an eq. point @ $\bar{x} = 0$ (i.e. if $\bar{x} \neq 0$ then apply change of coord.)
Then $\bar{x} = 0$ is stable iff for all $\varepsilon > 0$, there is $\delta_1 > 0$ ($\delta_1 < \varepsilon$) such that $\|x(0)\| < \delta_1 \Rightarrow \|x(t)\| < \varepsilon$ for all t

notation: $\|\cdot\|$: norm of a vector
(measure vector side)
ex: $\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$
$\|x\|_1 = |x_1| + \dots + |x_n|$



translation: start close (to eq. point)
stay close (to eq. point)