Modal decomposition of LTI systems

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State-space representation

$$\text{state equation:} \quad \dot{x}(t) \;=\; A\,x(t) \;+\; B\,d(t)$$

$$\text{output equation:} \quad y(t) \;=\; C\,x(t)$$

- **Solution to state equation**

$$x(t) \;=\; \mathrm{e}^{At}\,x(0) \;+\; \int_0^t \mathrm{e}^{A(t\,-\,\tau)}\,B\,d(\tau)\,\mathrm{d}\tau$$

$\mathrm{e}^{At}\,x(0)$ $\downarrow$ **unforced response**

$\int_0^t \mathrm{e}^{A(t-\tau)}\,B\,d(\tau)\,\mathrm{d}\tau$ $\downarrow$ **forced response**

Transform techniques

$$\dot{x}(t) \;=\; A\,x(t) \;+\; B\,d(t) \quad \xrightarrow{\text{Laplace transform}} \quad s\,\hat{x}(s) \;-\; x(0) \;=\; A\,\hat{x}(s) \;+\; B\,\hat{d}(s)$$

$$x(t) \;=\; \mathrm{e}^{At}\,x(0) \;+\; \int_0^t \mathrm{e}^{A(t-\tau)}\,B\,d(\tau)\,\mathrm{d}\tau$$

$$\Updownarrow$$

$$\hat{x}(s) \;=\; (sI \;-\; A)^{-1}\,x(0) \;+\; (sI \;-\; A)^{-1}\,B\,\hat{d}(s)$$

Natural and forced responses

- **Unforced response**

matrix exponential	resolvent
$x(t) \;=\; \mathrm{e}^{A\,t}\,x(0)$	$\hat{x}(s) \;=\; (sI \,-\, A)^{-1}\,x(0)$

- **Forced response**

impulse response	transfer function
$H(t) \;=\; C\,\mathrm{e}^{A\,t}\,B$	$H(s) \;=\; C\,(sI \,-\, A)^{-1}\,B$

⋆ **Response to arbitrary inputs**

$$y(t) \;=\; \int_0^t H(t \,-\, \tau)\,d(\tau)\,\mathrm{d}\tau \;\; \xrightarrow{\text{Laplace transform}} \;\; \hat{y}(s) \;=\; H(s)\,\hat{d}(s)$$

UNFORCED RESPONSES

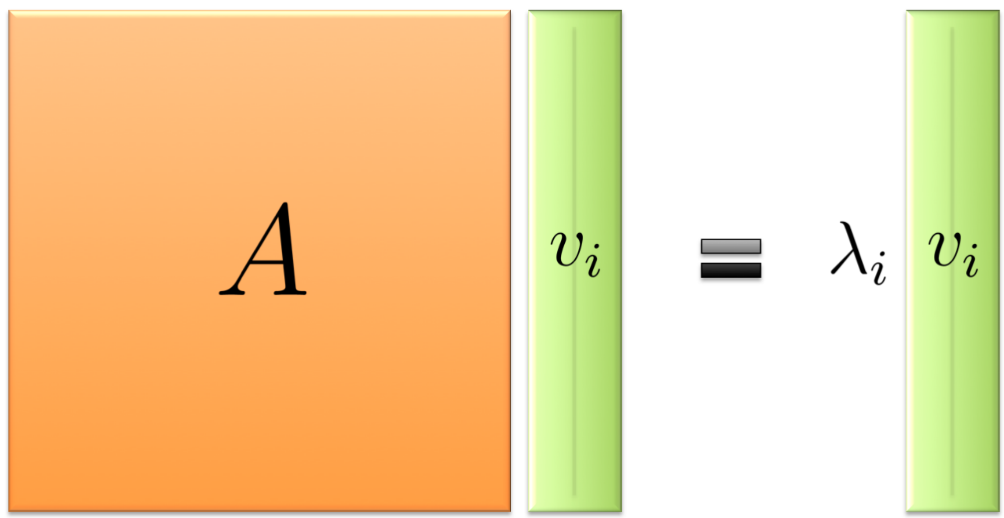
Systems with non-normal A

$$\dot{x}(t) \;=\; A\,x(t)$$

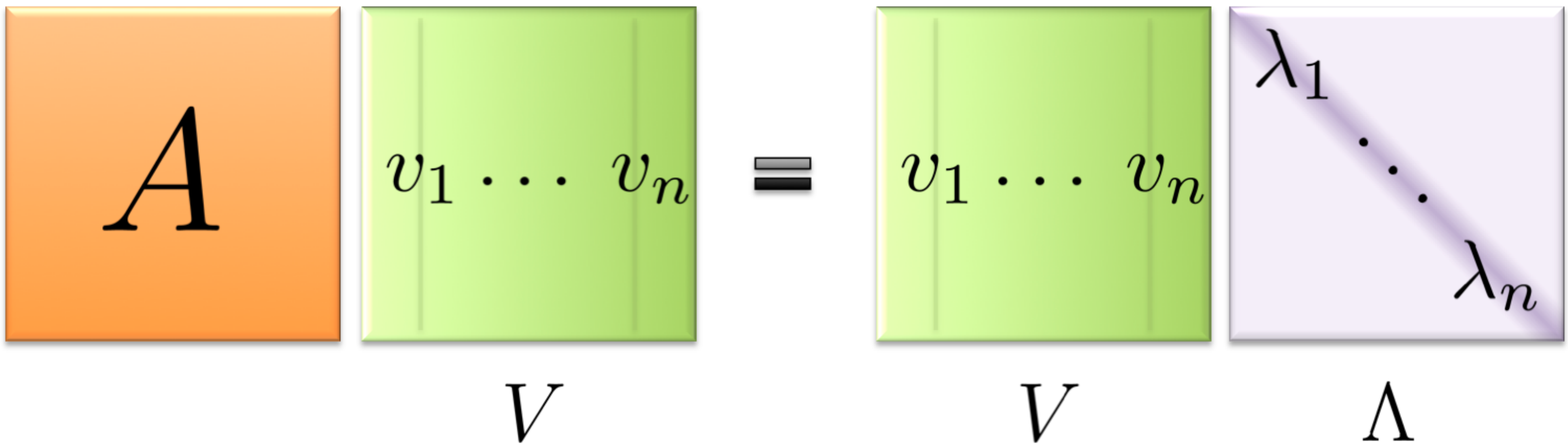
- **Non-normal operator: doesn't commute with its adjoint**

$$A\,A^* \;\neq\; A^*A$$

  - ⋆ **E-value decomposition of A**



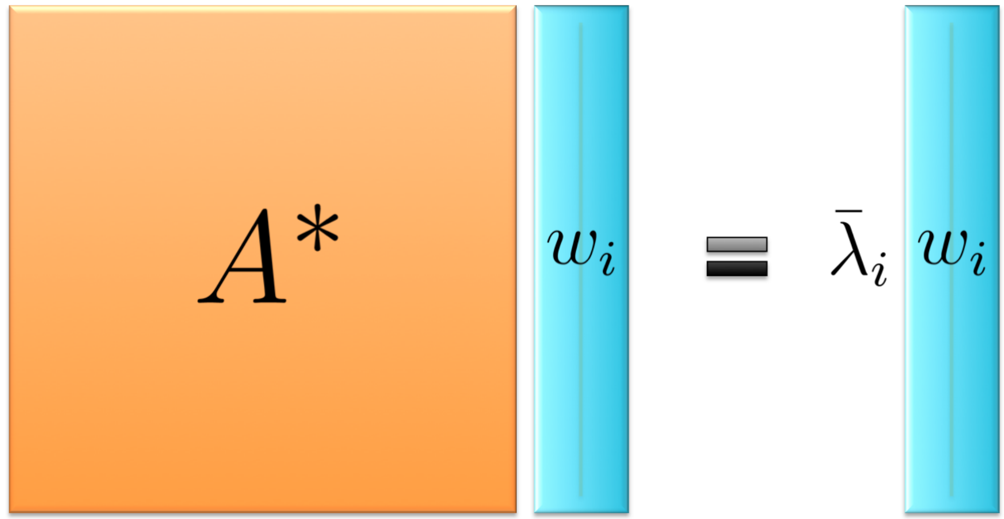
- **Let A have a full set of linearly independent e-vectors**



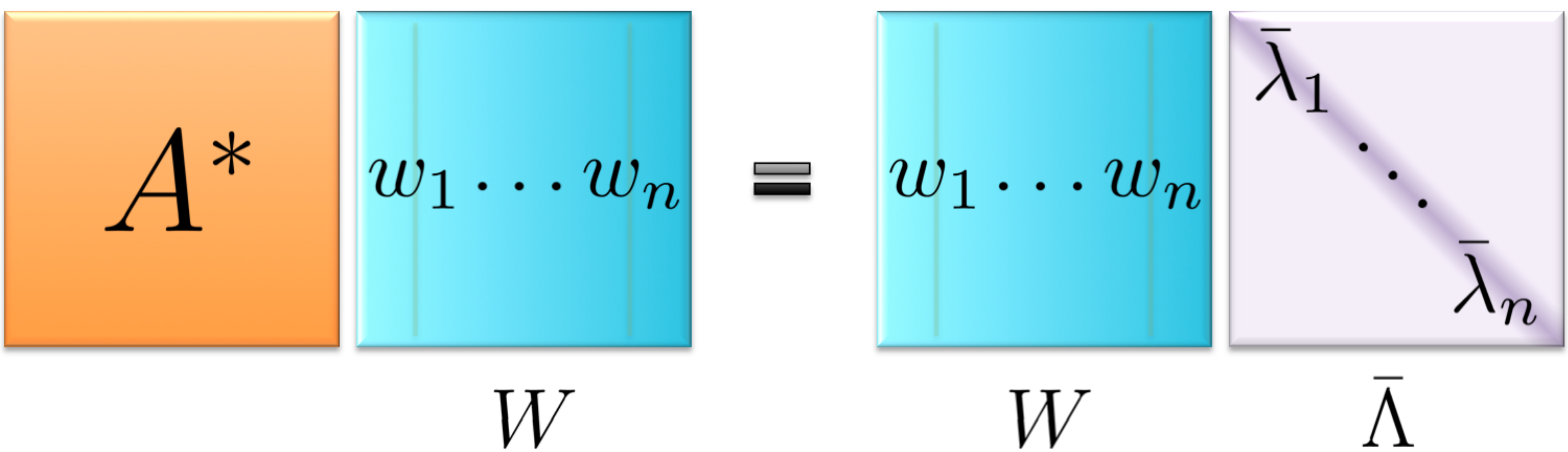
⋆ **normal A: unitarily diagonalizable**

$$A = V \Lambda V^*$$

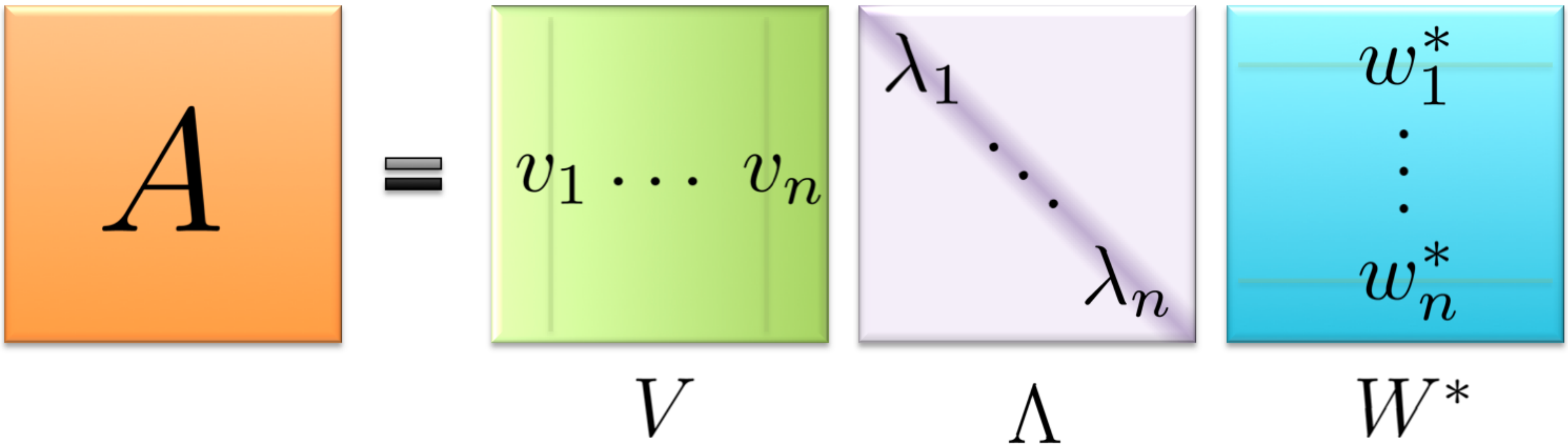
- **E-value decomposition of A^***



choose w_i such that $w_i^* v_j = \delta_{ij}$



- **Use V and W^* to diagonalize A**



⋆ **solution to** $\dot{x}(t) \;=\; A\,x(t)$

$$x(t) \;=\; \mathrm{e}^{A\,t}\,x(0) \;=\; \sum_{i\,=\,1}^{n} \mathrm{e}^{\lambda_i t}\, v_i\,(w_i{}^{*} x(0))$$

- **Right e-vectors**
  - ⋆ **identify initial conditions with simple responses**

$$x(t) \;=\; \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} \left({w_i}^* x(0) \right) v_i$$

$$\Big\downarrow \; x(0) \;=\; v_k$$

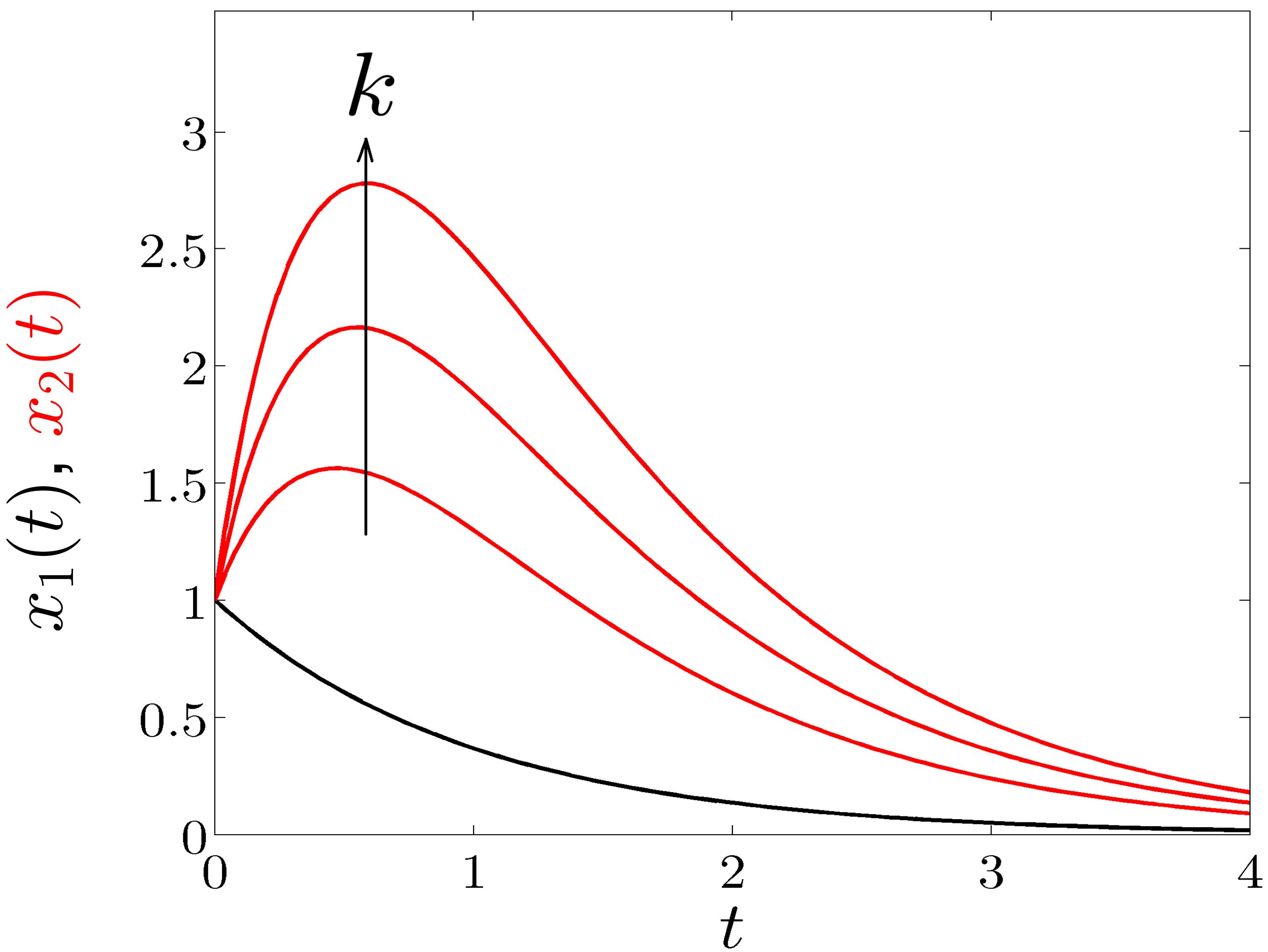
$$x(t) \;=\; \mathrm{e}^{\lambda_k t}\, v_k$$

- **Left e-vectors**
  - ⋆ **decompose state into modal components**

$$x(t) \;=\; \sum_{i\,=\,1}^{n} \mathrm{e}^{\lambda_i t} \left({w_i}^* x(0) \right) v_i \quad \xrightarrow{\text{i.p. with } w_k} \quad \left({w_k}^* x(t) \right) \;=\; \mathrm{e}^{\lambda_k t} \left({w_k}^* x(0) \right)$$

A toy example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & \mathbf{0} \\ k & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



- E-value decomposition of $A = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}$

$$\left\{ v_1 = \frac{1}{\sqrt{1 + k^2}} \begin{bmatrix} 1 \\ k \end{bmatrix},\; v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\left\{ w_1 = \begin{bmatrix} \sqrt{1 + k^2} \\ 0 \end{bmatrix},\; w_2 = \begin{bmatrix} -k \\ 1 \end{bmatrix} \right\}$$

solution to $\dot{x}(t) = A\,x(t)$:

$$x(t) = \left(\mathrm{e}^{-t} v_1 w_1^* + \mathrm{e}^{-2t} v_2 w_2^*\right) x(0)$$

$$\downarrow$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-t} x_1(0) \\ k\left(\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right) x_1(0) + \mathrm{e}^{-2t} x_2(0) \end{bmatrix}$$

- **E-values: misleading measures of transient response**

FORCED RESPONSES

(LATER IN THE COURSE)