

Modal decomposition of LTI systems

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State-space representation

state equation: $\dot{x}(t) = A x(t) + B d(t)$

output equation: $y(t) = C x(t)$

- Solution to state equation**

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B d(\tau) d\tau$$



**unforced
response**



**forced
response**

Transform techniques

$$\dot{x}(t) = Ax(t) + Bd(t) \quad \xrightarrow{\text{Laplace transform}} \quad s\hat{x}(s) - x(0) = A\hat{x}(s) + B\hat{d}(s)$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bd(\tau)d\tau$$



$$\hat{x}(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}B\hat{d}(s)$$

Natural and forced responses

- **Unforced response**

matrix exponential	resolvent
$x(t) = e^{At} x(0)$	$\hat{x}(s) = (sI - A)^{-1} x(0)$

- **Forced response**

impulse response	transfer function
$H(t) = C e^{At} B$	$H(s) = C (sI - A)^{-1} B$

- ★ **Response to arbitrary inputs**

$$y(t) = \int_0^t H(t - \tau) d(\tau) d\tau \xrightarrow{\text{Laplace transform}} \hat{y}(s) = H(s) \hat{d}(s)$$

UNFORCED RESPONSES

Systems with non-normal A

$$\dot{x}(t) = A x(t)$$

- **Non-normal operator: doesn't commute with its adjoint**

$$A A^* \neq A^* A$$

- ★ **E-value decomposition of A**

$$A v_i = \lambda_i v_i$$

- Let A have a full set of linearly independent e-vectors

$$A \begin{matrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{matrix} = \begin{matrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{matrix} \begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{matrix}$$

V V Λ

★ normal A : unitarily diagonalizable

$$A = V \Lambda V^*$$

- E-value decomposition of A^*

A diagram illustrating the E-value decomposition of a matrix A^* . On the left, an orange square represents the matrix A^* . To its right is a vertical cyan bar representing the weight vector w_i . An equals sign follows, and to the right is another vertical cyan bar representing the product $\bar{\lambda}_i w_i$.

choose w_i such that $w_i^* v_j = \delta_{ij}$

A diagram illustrating the full E-value decomposition of a matrix A^* . On the left, an orange square represents the matrix A^* . To its right is a cyan square representing the matrix W , with the columns labeled $w_1 \cdots w_n$. An equals sign follows, and to the right is another cyan square representing the matrix W , also with columns labeled $w_1 \cdots w_n$. To the right of this is a purple square representing the diagonal matrix $\bar{\Lambda}$, with the diagonal elements labeled $\bar{\lambda}_1$, \dots , and $\bar{\lambda}_n$.

- Use V and W^* to diagonalize A

$$A = V \Lambda W^*$$

V
 Λ
 W^*

★ solution to $\dot{x}(t) = Ax(t)$

$$x(t) = e^{At} x(0) = \sum_{i=1}^n e^{\lambda_i t} v_i (w_i^* x(0))$$

- **Right e-vectors**

- ★ **identify initial conditions with simple responses**

$$x(t) = \sum_{i=1}^n e^{\lambda_i t} (w_i^* x(0)) v_i$$
$$\downarrow x(0) = v_k$$
$$x(t) = e^{\lambda_k t} v_k$$

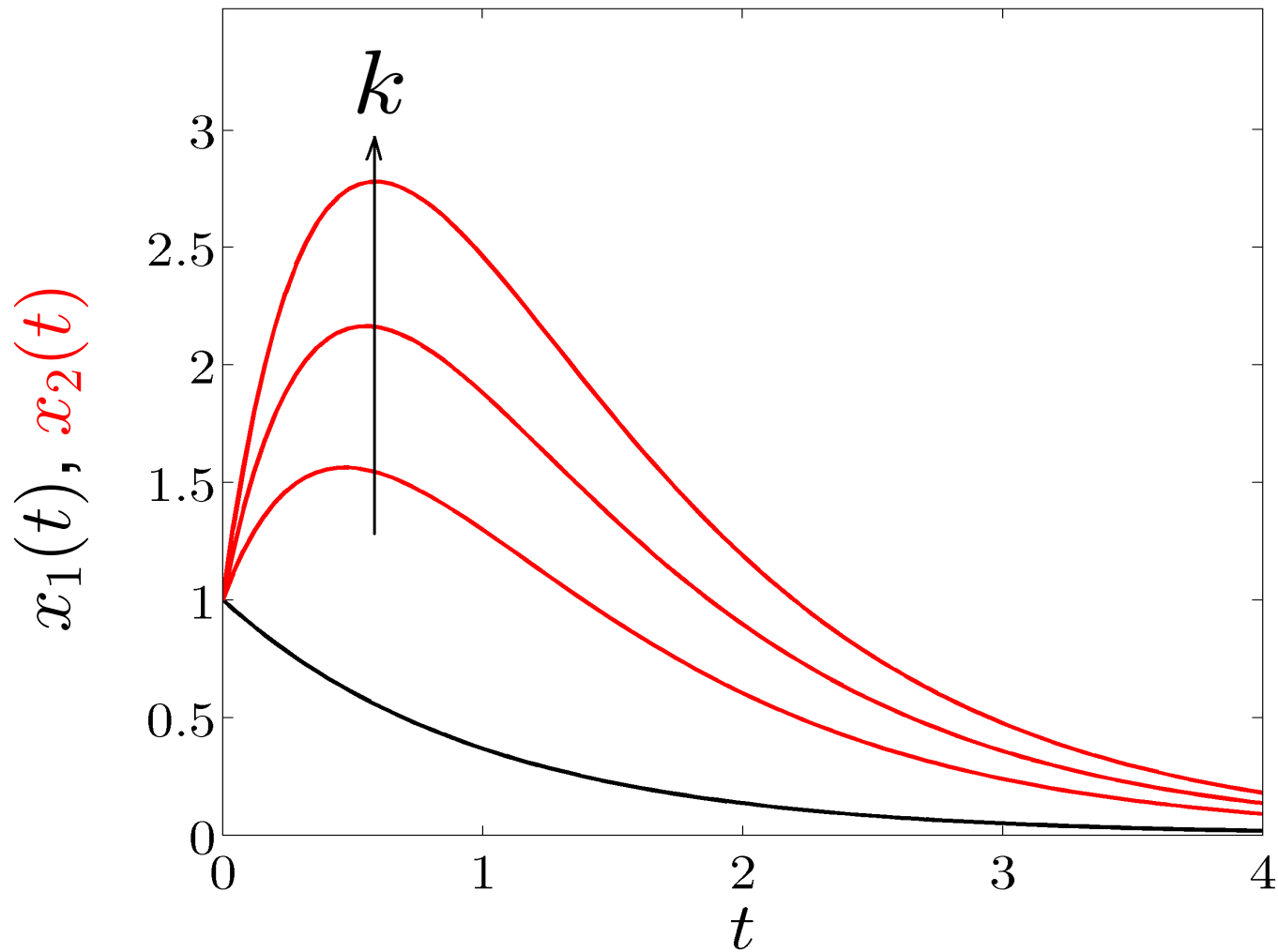
- **Left e-vectors**

- ★ **decompose state into modal components**

$$x(t) = \sum_{i=1}^n e^{\lambda_i t} (w_i^* x(0)) v_i \xrightarrow{\text{i.p. with } w_k} (w_k^* x(t)) = e^{\lambda_k t} (w_k^* x(0))$$

A toy example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & \mathbf{0} \\ k & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



- E-value decomposition of $A = \begin{bmatrix} -1 & 0 \\ k & -2 \end{bmatrix}$

$$\left\{ v_1 = \frac{1}{\sqrt{1+k^2}} \begin{bmatrix} 1 \\ k \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\left\{ w_1 = \begin{bmatrix} \sqrt{1+k^2} \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} -k \\ 1 \end{bmatrix} \right\}$$

solution to $\dot{x}(t) = Ax(t)$:

$$x(t) = (e^{-t} v_1 w_1^* + e^{-2t} v_2 w_2^*) x(0)$$

↓

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} x_1(0) \\ k(e^{-t} - e^{-2t}) x_1(0) + e^{-2t} x_2(0) \end{bmatrix}$$

- **E-values: misleading measures of transient response**

FORCED RESPONSES

(LATER IN THE COURSE)