9/19 Lecture 6

Last time: - Solutions to DT systems:

$$x(k) = \Phi(k,\ell)\, x(\ell) + \sum_{m=\ell}^{k-1} \Phi(k, m+1) \cdot B(m)\, u(m)$$

- In LTI case:

$$x(k) = A^k \cdot x(0) + \sum_{m=0}^{k-1} A^{k-m-1} B \cdot u(m)$$

- Z - transform
- Resolvent $R(z) = (zI - A)^{-1}$
- Transfer function

$$H(z) = C(zI - A)^{-1} B + D$$

Today: - Impulse + Frequency Response
- Same story for CT

Impulse Response

ex// single input u
single output y

LTI: $x_{k+1} = A x_k + B u_k$

$y_k = C x_k + D u_k$

$$u_k = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad \Big| \; x_0 = 0$$

$k = 0 \Rightarrow y_0 = C x_0 + D u_0 = D \cdot 1 = D$

$k = 1 \Rightarrow x_1 = A x_0 + B u_0 = A \cdot 0 + B \cdot 1 = B$

$y_1 = C \cdot x_1 + D \cdot u_1 = C \cdot B$

$k = 2 \Rightarrow x_2 = A \cdot x_1 + B \cdot u_1 = A x_1 = A \cdot B$

$y_2 = C \cdot x_2 = C \cdot A \cdot B$

$\Rightarrow y_k = C \cdot A^{k-1} \cdot B, \quad k \geq 2$ ... Markov Parameters

$\{y_0, y_1, y_2, \ldots, y_k, \ldots\} = \{D, CB, CAB, \ldots CA^{k-1}B\}$

$\hookrightarrow$ how system will respond

* Note: what we've derived follows directly from ☆ + the output equation:

$$y_k = Cx_k + Du_k = CA^k x_0 + \sum_{m=0}^{k-1} CA^{k-m-1} \cdot Bu_m + Du_k$$

set $x_0 = 0 \Rightarrow$

$$+ \; u_k = \left\{ \begin{array}{l} 1, \; k=0 \\ 0, \; k=1 \end{array} \right] \delta_k$$

$$y_k = \begin{cases} D & k=0 \\ CA^{k-1}B, & k \geq 0 \end{cases}$$

* Note: $Z\{\delta_k\} = \sum_{k=0}^{\infty} \delta_k z^{-k} = \delta_0 \cdot z^{-0} = \delta_0 = 1$

$$U(z) = 1 \Rightarrow Y(z) = H(z) \cdot U(z) = H(z) \Rightarrow T$$

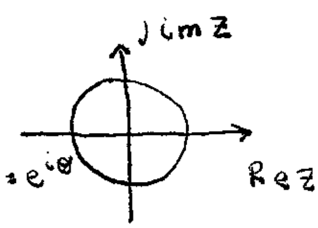
$$\boxed{\text{Transfer Function} = Z\left\{ \begin{array}{l} \text{impulse} \\ \text{response} \end{array} \right\}}$$

*note: In time domain:

$$y_k = \sum_{m=0}^{k-1} H(k-m) \cdot u(m)$$

$$y_k = \sum_{m=0}^{k-1} H_{k-m} \cdot u_m \qquad ; \; H(k-m) = CA^{k-m-1} \cdot B + D\delta_k$$

"Hit it with Z": $H(z) = C \cdot (zI - A)^{-1} \cdot B + D$



Frequency Response: $H(z)\big|_{z=e^{j\theta}} = H(e^{j\theta})$

same carries over to systems with many inputs + outputs
↳ called multivariable systems
"MIMO systems" vs "SISO systems"

ex// $y(k) \in \mathbb{R}^2$
$u(k) \in \mathbb{R}^3$
$Y(z) = H(z)\, U(z)$

$\rightarrow H_{ij}(z)$ determines mapping $U_j(z) \rightarrow Y_i(z)$

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} H_{11}(z) & \boxed{H_{12}(z)} & H_{13}(z) \\ H_{21}(z) & H_{22}(z) & H_{23}(z) \end{bmatrix} \begin{bmatrix} U_1(z) \\ U_2(z) \\ U_3(z) \end{bmatrix}$$

$$Y_i(z) = \sum_{j=1}^{m} H_{ij}(z)\, U_j(z) \; ; \; i = 1, \ldots, p$$

Solutions to Continuous Time Systems

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \quad \ldots \quad (1)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t) \quad \ldots \quad (2)$$
$$\hookrightarrow x(t_0) = x_0$$

Propose solution analogous to DT case:

$$x(t) = \phi(t, t_0)\cdot x(t_0) + \int_{t_0}^{t} \phi(t, \tau)\cdot B(\tau)\,u(\tau)\,d\tau \quad \ldots \quad (3)$$

τ - integration variable
- natural responce +
- forced response

Need to check 2 things:
1.) Initial conditions $x(t_0) = x_0$
2.) state equation (1)

1.) $t = t_0 \rightarrow 3$

$$x(t_0) = \phi(t_0, t_0)\cdot x(t_0) + \underbrace{\int_{t_0}^{t_0} \phi(t_0, \tau)\, B(\tau)\cdot u(\tau)\,d\tau}_{=0}$$

thus, if $\phi(t_0, t_0) = I$, then 1.) holds

2.) step 1: check if 2.) holds for $u \equiv 0$ (unforced system)

$x(t) = \phi(t, t_0)\,x_0(t_0)$ satisfies? $\dot{x}(t) = A(t)\,x(t)$

$$d(x(t))/dt = d\phi(t, t_0)/dt \cdot x(t_0) = A(t)\cdot\phi(t, t_0)\cdot x(t_0)$$

thus (3) satisfies (1) with $u \equiv 0$ if $\phi(t, t_0)$ solves:

$$d\phi(t, t_0)/dt = A(t)\,\phi(t, t_0) \rightarrow \phi(t, t_0) = I$$

recall: in DT: $\phi(k+1, l) = A(k)\,\phi(k, l) \Rightarrow \phi(l, l) = I$

step 2: $u \neq 0$, check if (3) satisfies 1.) (provided (1) holds

$$\dot{x}(t) = d(\phi(t, t_0))/dt \cdot x(t_0) + \phi(t, \tau)\cdot B(\tau)\cdot u(\tau)\big|_{\tau=t} + \int_{t_0}^{t} d\phi(t, \tau)/dt \cdot B(\tau)\,u(\tau)\,d\tau$$

$$\dot{x}(t) = A(t)\cdot\phi(t, t_0)\cdot x(t_0) + \overset{\phi(t,t)}{I}\cdot B(t)\cdot u(t) + \overset{(1)}{\int_{t_0}^{t}} A(t)\,\phi(t, \tau)\,B(\tau)\,u(\tau)\,d\tau$$
$$= A(t)\cdot x(t) + B(t)\,u(t)$$

yay!

LTI systems:

$\dot{x}(t) = Ax(t) + Bu(t)$

A, B constant matrices

$$\left.\begin{array}{l} d\phi(t,t_0)/dt = A\phi(t,t_0) \\ \phi(0) = I \end{array}\right\}$$ holds, but very stupid.

$\phi(t,t_0) = \phi(t-t_0)$

↳ only function of difference between t + t_0

⇒ can set $t_0 = 0$

$$x(t) = \phi(t) \cdot x(0) + \int_0^t \phi(t-\tau) B \cdot u(\tau)\, d\tau$$

next time: show that there is a "formula" for: $\phi(t)$