9/17 Lecture 5

last time:

- state transition matrix for DT systems
- $x(k+1) = A(k)\,x(k)$

  $x_\ell = x(\ell)$

$$x(\ell) \xrightarrow{\Phi(k,\ell)} x(k)$$

today:
- forced response
- z - transforms
- transfer function
- impulse response

objective: solve $x(k+1) = A(k)\,x(k) + B(k)\,u(k)$

$x(0) = 0$

(i.e. determine response to input u)

$k=0 \Rightarrow x(1) = A(0)\,\cancel{x(0)}^{0} + B(0)\,u(0)$

$k=1 \Rightarrow x(2) = A(1)\cdot x(1) + B(1)\cdot u(1) = A(1)B(0)u(0) + B(1)\cdot u(1)$

$k=2 \Rightarrow x(3) = A(2)A(1)B(0)u(0) + A(2)B(1)u(1) + B(2)u(2)$

$k=1: \quad \left[\, A(1)B(0) \;\vdots\; B(1) \,\right] \cdot \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}$

$k=2: \quad \left[\, A(2)A(1)B(0) \;\vdots\; A(2)B(1) \;\vdots\; B(2) \,\right] \begin{bmatrix} u(0) \\ u(1) \\ u(2) \end{bmatrix}$

Alternatively, we can write forced response as:

$$x_f(k) = \sum_{m=\ell}^{k-1} \Phi(k, m+1)\,B(m)\,u(m)$$

ex// $k=2,\ \ell=0$

$$\underbrace{\Phi(2,1)}_{A(1)} B(0)\,u(0) + \underbrace{\Phi(2,2)}_{I} B(1)\cdot u(1)$$

recall $\Phi(k,\ell) = A(k-1)\cdots A(\ell)$

initial time

thus: $x(k) = \underbrace{\Phi(k,\ell)\cdot x(\ell)}_{\text{unforced response}} + \underbrace{\sum \Phi(k, m+1)\, B(m)\cdot U(m)}_{\text{forced response}}$

For time invariant system:

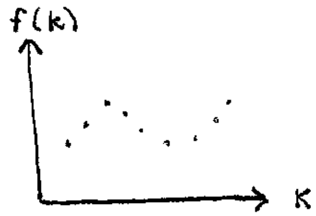
$A(k) = A = \text{constant}$

$B(k) = B = \text{constant}$

instead of $\phi(k, m+1)$

$X(k) = \underbrace{A^k \cdot x(0)}_{\phi(k-m-1)} + \sum_{m=0}^{k-1} A^{k-m-1}\, B \cdot U(m)$

Z - Transform



$\xrightarrow{Z}$ $F(z)$

$z \in \mathbb{C}$

$\{f_0, f_1, \ldots\} \xrightarrow[z\in\mathbb{C}]{Z} F(z)$

$f(0), f(1), \ldots$

one-sided Z transform

$F(z) = \sum_{k=0}^{\infty} f_k \cdot z^{-k} = Z(f)$

Properties of Z - transform

1.) Linearity

$\Rightarrow \{f\}, \{g\}$ (2 sequences)

α, β (2 scalars)

$Z\{\alpha f + \beta g\} = \alpha Z\{f\} + \beta Z\{g\} = \alpha F(z) + \beta G(z)$

follows from linearity of sumation Σ

2.) Shift

a.) $g_k := f_{k-1}$

$Z\{g\} = \sum_{k=0}^{\infty} g_k \cdot z^{-k} = \sum_{k=0}^{\infty} f_{k-1} z^{-k}$ $\quad \dfrac{n = k-1}{k = n+1}$

$= \sum_{n=-1}^{\infty} f_n z^{-(n+1)}$

$= z^{-1}\{f_{-1} z^{-(-1)} + \sum_{n=0}^{\infty} f_n z^{-n}\} \Rightarrow$

$G(z) = z^{-1}(F(z))$

thus:

$\xrightarrow{F(z)} \boxed{z^{-1}} \xrightarrow{G(z)}$

one step delay

Properties continued...

2.) Shift

b.) $g_k = f_{k+1}$

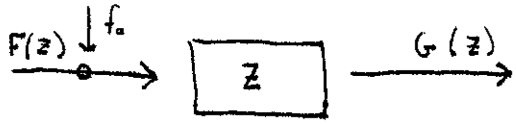
$$Z\{g\} = \sum_{k=0}^{\infty} g_k z^{-k} = \sum_{k=0}^{\infty} f_{k+1} z^{-k} \qquad \frac{n = k+1}{k = n-1}$$

$$= \sum_{n=1}^{\infty} f_n \cdot z^{-(n-1)}$$

$$= z \cdot \left\{ -f_0 + \sum_{n=0}^{\infty} f_n z^{-n} \right\} \Rightarrow$$

$$G(z) = z\,(F(z) - f_0)$$

thus:



one-step forward

3.) Converts convolution to multiplication

$$y(k) = \sum_{i=0}^{k} H(k-i) \cdot u(i)$$

$$\downarrow Z$$

$$Y(z) = H(z) \cdot U(z)$$

$$\hookrightarrow H(z) = Z\{H_0, H_1, \dots\}$$

Back to: $x_{k+1} = A x_k + B u_k \quad \dots \quad (1)$

$y_k = C x_k + D u_k \quad \dots \quad (2)$

i.c. $x(0) = x_0$

$Z(1) \Rightarrow$ from 2b.

$$z \cdot \{x(z) - x_0\} = A \cdot x(z) + B \cdot U(z)$$

$$z\,x(z) - A(x(z)) = z \cdot x_0 + B \cdot U(z)$$

eye(n)

$$(z I - A)\,x(z) = z \cdot x_0 + B \cdot U(z)$$

$$\boxed{x(z) = z \cdot (zI - A)^{-1} x_0 + (zI - A)^{-1} B \cdot U(z) \quad \dots \quad (3)}$$

$$Z(2) \Rightarrow C \cdot x(z) + D \cdot U(z) \dots \quad (4)$$

$$(3) \rightarrow (4) \Rightarrow Y(z) = z \cdot C \cdot \underbrace{(zI - A)}_{\text{resolvent of matrix } A}{}^{-1} \cdot x_0 + \underbrace{[C \cdot (zI - A)^{-1} B + D]}_{\text{transfer function}} U(z)$$

$$R(z) = (zI - A)^{-1}$$

note $A^k = Z^{-1}\{z \cdot R(z)\}$

transfer function $H(z)$: mapping from $\underset{\text{input}}{U(z)}$ to $\underset{\text{output}}{Y(z)}$

$\hookrightarrow Y(z) = H(z)\,U(z)\,\big|_{x_0 = 0}$

$Z \mid x(k) = A^k x_0$

$x(z) = Z\{A^k\}\,x_0 = z\,R(z) \cdot x_0$

$\Rightarrow Z\{A^k\} = z \cdot R(z)$

If a was a scalar...

$$\sum_{k=0}^{\infty} \left(\frac{z}{a}\right)^{-k} = \sum_{k=0}^{\infty} q^{-k} = \frac{1}{1-q^{-1}} \qquad |q| < 1$$

$\rightarrow$ series (Heumann Series)

$$\sum_{k=0}^{\infty} M^k = (I - M)^{-1} \quad \text{provided } \|M\| < 1$$