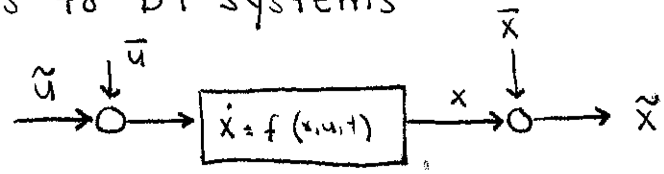
# 9/12 Lecture 4

last time: state-space model
equilibrium points
linearization

today: linearization (example)
solutions to DT systems

$$\left.\begin{aligned} \dot{x} &= f(x, y, t) \\ y &= \bar{g}(x, y, t) \end{aligned}\right\}$$




for "small" $\|\tilde{x}\|$, $\|\tilde{u}\|$, $\|\tilde{y}\|$

$$\dot{\tilde{x}} = \frac{\partial f}{\partial x}\Big|_{(\bar{x},\bar{u})} \cdot \tilde{x} + \frac{\partial f}{\partial u}\Big|_{(\bar{x},\bar{u})} \tilde{u}$$

similarly for $y = g(x, y, t) \rightarrow \tilde{y} = \frac{\partial g}{\partial x}\Big|_{(\bar{x},\bar{u})} \cdot \tilde{x} + \frac{\partial g}{\partial u}\Big|_{\bar{x},\bar{u}} \cdot \tilde{u}$

ex// Inverted Pendulum (see lecture #3 9/10)

$$f(x, u) = \begin{bmatrix} x_2 \\ \sin(x_1) + u \end{bmatrix}$$

$g(x, u) = x_1$

we showed that for $\bar{u} = 0$

$$\bar{x}_{up} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \quad \bar{x}_{down} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$\text{* Jacobian: } \frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \cos(x_1) & 0 \end{bmatrix}$$

$$\frac{\partial F}{\partial u} = \begin{bmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \partial g/\partial x_1 & \partial g/\partial x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$\partial g/\partial u = 0$

we need to evaluate these functions around $(\bar{x}, \bar{u})$

ex// continued...

$$A_{up} = \frac{dF}{dx}\Big|_{(\bar{x}_{up},0)} = \begin{bmatrix} 0 & 1 \\ \cos(0) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{down} = \frac{df}{dx}\Big|_{(\bar{x}_{down},0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

consider $\bar{x}_{45°} = \begin{bmatrix} \pi/4 \\ \boxed{0} \end{bmatrix}$ Q: what value should $\bar{u}$ have?

from $\dot{x}_1 = \dot{x}_2$

$\dot{\bar{x}}_2 = \sin(x_1) + \bar{u} \Rightarrow$

$\bar{u} = -\sin(\bar{x}_1) = -\sqrt{2}/2$

$$A_{45°} = \begin{bmatrix} 0 & 1 \\ \cos \pi/4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sqrt{2}/2 & 0 \end{bmatrix}$$

* note: linearization around $(\bar{x}(t), \bar{u}(t))$ can yield a time varying model if if the original system is time-invarient!

$(i.e.\ f = f(x,u)\ ;\ g = g(x,u))$

for invert. pend. linearization

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ \cos(\bar{x}_1(t)) & 0 \end{bmatrix}}^{A(t)} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} \tilde{u}$$

$$\tilde{y} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \underbrace{0}_{D} \cdot \tilde{u}$$

aside: $\dot{\bar{x}} + \dot{\tilde{x}} = f(\bar{x} + \tilde{x}, \bar{u} + \tilde{u}) = f(\bar{x}, \bar{u}) + \frac{df}{dx}\Big|_{(\bar{x},\bar{u})} \tilde{x} + \frac{df}{du}\Big|_{(\bar{x},\bar{u})} \tilde{u}$

note, if $(\bar{x}, \bar{u})$ is ~~what~~ solution to state eq $\dot{\bar{x}} = f(\bar{x}, \bar{u})$ so $f(\bar{x}, \bar{u})$ cancels.

Solutions to discrete time systems

state-space model:

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k) \quad \ldots \quad (1)$$

$$y(k) = C(k)\,x(k) + D(k)\,u(k) \quad \ldots \quad (2)$$

↳ initial conditions $x(k_0) = x_0 \quad \ldots \quad (3)$

want to solve (1):

$x(k) = $ (natural response) + (forced response)

↳ or unforced / zero-input ↳ caused by $x_0$

↳ caused by $u$

natural response

$$\left.\begin{aligned} x(k+1) &= A(k)\,x(k) \\ x(0) &= x_0 \end{aligned}\right\} \; k = 0, 1, 2, \ldots$$

plug + chug

$k=0$: $x(0+1) = A(0) \cdot x(0)$

$x(1) = A(0)\,x_0$

$k=1$: $x(2) = A(1) \cdot x(1)$

$x(2) = A(1) \cdot A(0) \cdot x_0$

$\vdots$

$$x(k) = A(k-1)\,A(k-2) \ldots A(1)\,A(0) \cdot x_0$$

$$= \phi(k, 0)$$

if instead we had:

$$x(k+1) = A(k)\,x(k)$$

$$x(l) = x_l$$

$$\Rightarrow x(k) = A(k-1) \ldots A(l+1)\,A(l) \cdot x_l$$

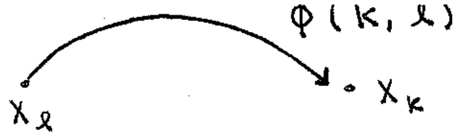
$$= \phi(k, l)$$

↳ initial time

↳ final time

$\phi(k, l) =$ state-transition matrix

(matrix valued function of 2 arguements)

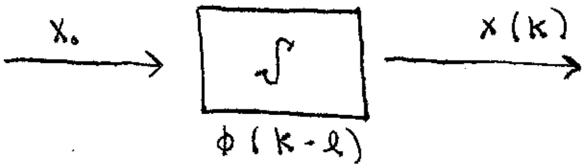



$\leftarrow$ state trans matrix

If our system was time-invariant:

$x(k+1) = \underbrace{A}_{\text{constant}} \cdot x(k)$

$\phi(k,\ell) = \phi(k-\ell) = A^{k-\ell}$

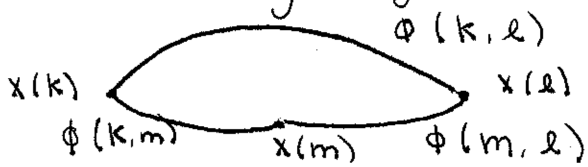



input: any excitation that is entering the system

Properties of $\phi(k,\ell)$

1.) $\phi(\ell,\ell) = \underbrace{I}_{\text{Identity matrix}}$ $\quad (x(\ell) = \phi(\ell,\ell)\cdot x(\ell))$

2.) "connecting flight"




$\phi(k,\ell) = \phi(k,m)\cdot\phi(m,\ell)$

3.) if instead we had $x(k+1) = A(k)\,x(k)$

$\phi(k+1,\ell) = A(k)\,A(k-1)\ldots A(\ell) = A(k)\,\phi(k,\ell)$

$\phi(\ell,\ell) = I$

these properties have similar parts in cont. time

Coming soon... in continuous time

$$\frac{\partial\,\phi(t,\tau)}{dt} = A(t)\cdot\phi(t,\tau)$$

$$\phi(\tau,\tau) = I$$

other 2 properties hold.

ex// $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \phi(0) = I$

$\phi(1) = A$

$\phi(2) = A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

in CT, $\phi(t,\tau)$ is always invertable

Forced Responses

$$x(k+1) = A(k)\,x(k) + B(k)\cdot u(k)$$

$$x(0) = 0$$

$$k=0 \Rightarrow x(1) = A(0)\cdot 0 + B(0)\cdot u(0)$$
$$= B(0)\cdot u(0)$$

$$k=1 \Rightarrow x(2) = A(1)\cdot x(1) + B(1)\cdot u(1)$$
$$= A(1)\cdot B(0)\cdot u(0) + B(1)\,(u(1))$$
$$= \left[ A(1)\cdot B(0) \;\vdots\; B(1) \right] \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}$$

$$k=2 \Rightarrow x(3) = A(2)\,x(2) + B(2)\,x(2)$$
$$= \left[ A(2)\cdot A(1)\cdot B(0) \;\vdots\; A(2)\cdot B(1) \;\vdots\; B(2) \right] \begin{bmatrix} u(1) \\ u(2) \\ u(3) \end{bmatrix}$$