EE/AEM 5231 HW 1 Fall 2013

Due Friday 09/13/13 (5pm; Xiaofan's office, Keller Hall 5-149)

1. Problem 2.2 from the book (page 20; attached).

2. Consider the unforced mass-spring system

$$m\,\ddot{y} \,+\, g(y) \,=\, 0$$

with three different models for the spring force

- **hardening spring:** $g(y) \,=\, k\left(1 \,+\, y^2\right) y;$
- **softening spring:** $g(y) \,=\, k\left(1 \,-\, y^2\right) y;$
- **linear spring:** $g(y) \,=\, k\,y,$

and $k > 0$.

(a) Determine a state-space representation of this system.

(b) Find equilibrium points of the above systems. Discuss your observations for three different spring force models.

(c) Is this system

- causal,
- time-varying,
- linear,
- memoryless,
- finite-dimensional?

Explain.

(d) For three different spring force models with $m = k = 1$, use Matlab to simulate systems' responses from different initial conditions. Plot corresponding results in the phase plane (horizontal axis determined by position $y(t)$, vertical axis determined by velocity $\dot{y}(t)$) and discuss your observations.

3. The system shown in Figure 1 is composed of a first order system followed by a saturation element. Which of the following properties does this system have a) causality, b) linearity c) time-invariance? Is the system memoryless? Compute the output y that corresponds to the periodic input in Figure 1.

Note: The saturation function works as follows: if the two signals g and y are related by $y(t) =$ Saturation $(g(t))$, then

$$y(t) \;=\; \begin{cases} g(t) & \text{if} \quad |g(t)| \;\leq\; 1 \\ 1 & \text{if} \quad g(t) \;>\; 1 \\ -1 & \text{if} \quad g(t) \;<\; -1 \end{cases}$$

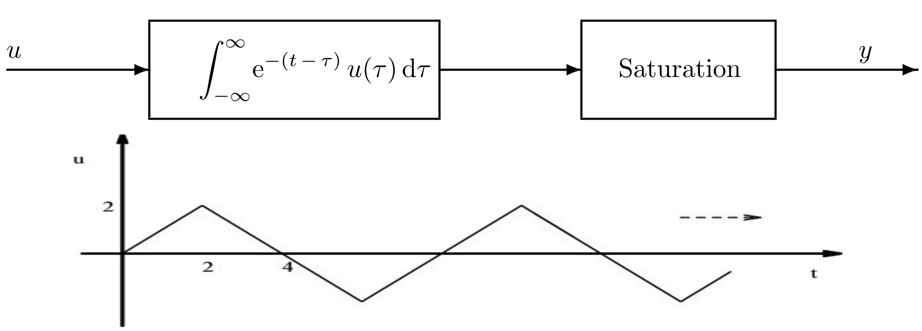

Figure 1: System in Problem 3.

Does such an equilibrium point always exist?

(d) Assume that $b = 1/2$ and $mg\ell = 1/4$. Compute the torque $T(t)$ needed for the pendulum to fall from $\theta(0) = 0$ with constant velocity $\dot{\theta}(t) = 1, \forall t \geq 0$. Linearize the system around this trajectory. □

2.2 (Local linearization around a trajectory). A single-wheel cart (unicycle) moving on the plane with linear velocity v and angular velocity ω can be modeled by the nonlinear system

$$\dot{p}_x = v\cos\theta, \qquad \dot{p}_y = v\sin\theta, \qquad \dot{\theta} = \omega, \tag{2.11}$$

where (p_x, p_y) denote the Cartesian coordinates of the wheel and θ its orientation. Regard this as a system with input $u := \begin{bmatrix} v & \omega \end{bmatrix}' \in \mathbb{R}^2$.

(a) Construct a state-space model for this system with state

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} := \begin{bmatrix} p_x\cos\theta + (p_y - 1)\sin\theta \\ -p_x\sin\theta + (p_y - 1)\cos\theta \\ \theta \end{bmatrix}$$

and output $y := \begin{bmatrix} x_1 & x_2 \end{bmatrix}' \in \mathbb{R}^2$.

(b) Compute a local linearization for this system around the equilibrium point $x^{\text{eq}} = 0, u^{\text{eq}} = 0$.

(c) Show that $\omega(t) = v(t) = 1$, $p_x(t) = \sin t$, $p_y(t) = 1 - \cos t$, $\theta(t) = t$, $\forall t \geq 0$ is a solution to the system.

(d) Show that a local linearization of the system around this trajectory results in an LTI system. □

Attention! Writing the system in the carefully chosen coordinates x_1, x_2, x_3 is crucial to getting an LTI linearization. If one tried to linearize this system in the original coordinates p_x, p_y, θ with dynamics given by (2.11), one would get an LTV system.

2.3 (Feedback linearization controller). Consider the inverted pendulum in Figure 2.6.

(a) Assume that you can directly control the system in torque, i.e., that the control input is $u = T$.

Design a feedback linearization controller to drive the pendulum to the upright position. Use the following values for the parameters: $\ell = 1\,\text{m}$, $m = 1\,\text{kg}$, $b = 0.1\,\text{N}\,\text{m}^{-1}\,\text{s}^{-1}$, and $g = 9.8\,\text{m}\,\text{s}^{-2}$. Verify the performance of your system in the presence of measurement noise using Simulink®.

(b) Assume now that the pendulum is mounted on a cart and that you can control the cart's jerk, which is the derivative of its acceleration a. In this case,

$$T = -m\,\ell\,a\cos\theta, \qquad \dot{a} = u.$$

Design a feedback linearization controller for the new system.

What happens around $\theta = \pm\pi/2$?

Note that, unfortunately, the pendulum needs to pass by one of these points for a swing-up, i.e., the motion from $\theta = \pi$ (pendulum down) to $\theta = 0$ (pendulum upright). □