Does Alternating Direction Method of Multipliers Converge for Nonconvex Problems?

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The Main Content

The Alternating Direction Method of Multipliers (ADMM) is a very popular method for dealing with large-scale optimization problems.

Applications to classical problems
1. LP [Boyd 11], [Ye 15]
2. SDP [Wen-Goldfarb-Yin 10], [Sun-Toh-Yang 15]
3. QCQP [Huang-Sidiropoulos 16]

Applications to emerging areas
2. Training neural networks [Taylor et al 16]
3. Smart grid [Dall’Anese et al 13], [Peng-Low 15]
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Research Question

Q: Is ADMM convergent for nonconvex problems?

A: Yes, for global consensus and sharing problems, and many more
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- Develop a new framework for analyzing the nonconvex ADMM
- Obtain key insights on the behavior of the algorithm
- Motivate new research in theory and applications
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Impact

This Work

New Connections

SAG
[Defazio et al 14]

SAGA
[Le Roux et al 12]

IAG
[Blatt et al 07]

EXTRA
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New Analysis

[Kumar et al 16]

[Yang-Pong-Chen 15]

New Applications

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Outline

1 Overview

2 Literature Review

3 A New Analysis Framework
   - A Toy Example
   - Nonconvex Consensus Problem
   - Algorithm and Analysis

4 Recent Advances

5 Conclusion
The Basic Setup

Consider the following problem with $K$ blocks of variables $\{x_k\}_{k=1}^K$:

$$\min f(x) := \sum_{k=1}^{K} h_k(x_k) + g(x_1, \cdots, x_K) \quad (P)$$

s.t. $\sum_{k=1}^{K} A_k x_k = q$, $x_k \in X_k$, $\forall k = 1, \cdots, K$

- $h_k(\cdot)$: a convex nonsmooth function
- $g(\cdot)$: a smooth, possibly nonconvex function
- $Ax = q$: linearly coupling constraint, $A_k \in \mathbb{R}^{M \times N_k}$, $q \in \mathbb{R}^M$
- $X_k \subseteq \mathbb{R}^{N_k}$: a closed convex set
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The augmented Lagrangian (AL) is given by

$$L(x; y) = \sum_{k=1}^{K} h_k(x_k) + g(x_1, \cdots, x_K) + \langle y, q - Ax \rangle + \frac{\rho}{2} \|q - Ax\|^2,$$

where $\rho > 0$ is the penalty parameter; $y$ is the dual variable.
The Basic Setup

- The ADMM performs a block coordinate descent (BCD) on the AL, followed by an (approximate) dual ascent
- Inexactly optimizing the AL often yields closed-form solutions

### The ADMM Algorithm
At each iteration $t + 1$:

**Update the primal variables:**

$$x_k^{t+1} = \arg \min_{x_k \in X_k} L(x_1^{t+1}, \ldots, x_{k-1}^{t+1}, x_k, x_{k+1}, \ldots, x_K^t; y^t), \forall k.$$ 

**Update the dual variable:**

$$y^{t+1} = y^t + \rho (q - Ax^{t+1}).$$
The Convex Case

ADMM works for **convex, separable, 2-block** problems.

The $g(\cdot)$ and $h_k(\cdot)$’s convex; $g(x_1, \cdots, x_k) = \sum_{k=1}^{K} g_k(x_k); \ K = 2$

- Many classic works on the analysis [Glowinski-Marroco 75], [Gabay-Mercier 76] [Glowinski 83]...
- Equivalence to Douglas-Rachford Splitting and PPA [Gabay 83], [Eckstein-Bertsekas 92]
- Convergence rates and iteration complexity analysis [Eckstein 89] [He-Yuan 12] [Deng-Yin 12] [Hong-Luo 12]
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Solving Nonconvex Problems?

- All the works mentioned before are for **convex** problems
- Recently, widely (and wildly) applied to **nonconvex** problems as well
  - Distributed clustering [Forero-Cano-Giannakis 11]
  - Matrix separation/completion [Xu-Yin-Wen-Zhang 11]
  - Phase retrieval [Wen-Yang-Liu-Marchesini 12]
  - Distributed matrix factorization [Ling-Yin-Wen 12]
  - Manifold optimization [Lai-Osher 12]
  - Asset allocation [Wen-Peng-Liu-Bai-Sun 13]
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Application 1: Nonnegative Tensor Factorization

Figure: ADMM for solving tensor factorization problem [Liavas-Sidiropoulos 14]
Issues and Challenges

- **Pros:** Nonconvex ADMM achieves excellent numerical performance

- **Cons:** A general lack of global performance analysis

  *Convergence claim*

  But this is a big "IF"!
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**Convergence claim**

1. IF the successive differences of all the primal and dual variables go to zero (e.g., $x^{t+1} - x^t \to 0, y^{t+1} - y^t \to 0$)

2. Then any limit point is a stationary solution

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Issues and Challenges

- The assumption on iterates is **uncheckable a priori**

- "Assume" (without proving) that feasibility holds in the limit

- An exception [Zhang 10]: convergence for certain special QP
  1. The AL is strongly convex
  2. Only has the linear constraint
  3. The dual stepsize is very small
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Issues and Challenges

Rigorously analyzing nonconvex ADMM is challenging

- **Cases I**: Without the linear constraint, reduces to the classic BCD
  - Can diverge for general nonconvex $g(\cdot)$ with $K \geq 3$ [Powell 73]

- **Cases II**: With the linear constraint and $K = 1$
  - Can diverge for any fixed $\rho > 0$ [Wang-Yin-Zeng 16]
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5. Conclusion
A Toy Example

First consider the following toy nonconvex example

$$\min_{x,z} \frac{1}{2} x^T A x + b z, \quad \text{s.t.} \quad z \in [1, 2], \ z = x$$

where $A$ is a symmetric matrix; $x \in \mathbb{R}^N$
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ADMM Convergent?
A Toy Example (cont.)

- Randomly generate the data matrices $A$ and $b$ with $N = 10$

- Plot the following
  1. Primal feasibility gap: $||z - x||$
  2. The optimality measure: $||x - \text{proj}[x - (Ax + b)]||$
  3. The $x$-feasibility gap: $||x - \text{proj}(x)||$

- All three quantities go to zero iff a stationary solution has been reached
First Try: $\rho = 20$

![Graph showing the ADMM feasibility measure, original feasibility measure, and optimality measure for $N = 10$, $\rho = 20$.](image)

**Figure:** $N = 10$, $\rho = 20$
Second Try: $\rho = 200$

Figure: $N = 10, \rho = 200$
A Toy Example (cont.)

Figure: $n = 10, \rho = 20$

Figure: $n = 10, \rho = 200$
The convergence is $\rho$-dependent

When $\rho$ is small, the algorithm fails to converge

Different from the convex case, where any $\rho > 0$ should work

Reminiscent to the AL method, careful choice of $\rho$ in nonconvex case
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Consider a nonconvex global consensus problem

A distributed optimization problem defined over a network of $K$ agents
Problem Setup: A Nonconvex Consensus Problem

- Consider a nonconvex global consensus problem
- A distributed optimization problem defined over a network of $K$ agents
- Formally, the problem is given by

$$\min \sum_{k=1}^{K} g_k(x_k) + h(x_0), \quad \text{s.t.} \quad x_k = x_0, \forall k = 1, \cdots, K, x_0 \in X.$$
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Problem Setup: A Nonconvex Consensus Problem

- Wide applications in distributed signal and information processing, parallel optimization, etc [Boyd et al 11]
- For example, in the distributed sparse PCA problem [H.-Luo-Razaviyayn 14]
  \[ g_k(x_k) = -x_k^T A_k^T A_k x_k : A_k^T A_k \text{ is the covariance matrix for local data} \]
  \[ h(\cdot) : \text{some sparsity promoting nonsmooth regularizer} \]
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2. $h(\cdot)$: some sparsity promoting nonsmooth regularizer
The Algorithm

- The AL function is given by

\[ L(\{x_k\}, x_0; y) = \sum_{k=1}^{K} g_k(x_k) + h(x_0) + \sum_{k=1}^{K} \langle y_k, x_k - x_0 \rangle + \sum_{k=1}^{K} \frac{\rho_k}{2} \| x_k - x_0 \|^2. \]

**Algorithm 1. The Consensus ADMM**

At each iteration \( t + 1 \), compute:

\[ x_0^{t+1} = \arg\min_{x_0 \in X} L(\{x_k^t\}, x_0; y^t). \]

Each node \( k \) computes \( x_k \) by solving:

\[ x_k^{t+1} = \arg\min_{x_k} g_k(x_k) + \langle y_k^t, x_k - x_0^{t+1} \rangle + \frac{\rho_k}{2} \| x_k - x_0^{t+1} \|^2. \]

Each node \( k \) updates the dual variable:

\[ y_k^{t+1} = y_k^t + \rho_k (x_k^{t+1} - x_0^{t+1}). \]
Illustration: $x_0$ update

$x_0$ solves: $x_0^{t+1} = \arg\min_{x_0 \in X} L(\{x_k^t\}, x_0; y^t)$ (often with closed-form)
Illustration: broadcast

Broadcasts the most recent $x_0$

Diagram:
- $x_0$
- $x_1, y_1$
- $x_2, y_2$
- $x_3, y_3$
- $x_4, y_4$
- $x_K, y_K$
Illustration: \((x_k, \lambda_k)\) update

\[
x_k \text{ solves: } x_{k+1} = \arg \min_{x_k} g_k(x_k) + \langle y_k^t, x_k - x_0^{t+1} \rangle + \frac{\rho_k}{2} \| x_k - x_0^{t+1} \|^2.
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Illustration: aggregate

Aggregate \((x_k, y_k)\) to the central node
Main Assumptions

Assumption A

A1. Each $g_k$ has Lipschitz continuous gradient:

$$\|\nabla_k g_k(x_k) - \nabla_k g_k(z_k)\| \leq L_k \|x_k - z_k\|, \forall x_k, z_k, k = 1, \ldots, K.$$  

Moreover, $h$ is convex (possible nonsmooth); $X$ is a closed convex set.

A2. $\rho_k$ is large enough such that:

1. For all $k$, the $x_k$ subproblem is strongly convex with modulus $\gamma_k(\rho_k)$;
2. For all $k$, the following is satisfied

$$\rho_k > \max \left\{ \frac{2L_k^2}{\gamma_k(\rho_k)}, L_k \right\}.$$  

A3. $f(x)$ is bounded from below over $X$.  

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Proof Ideas: Preliminary

- **Question:** Can we leverage the existing analysis for the convex case?

- Unfortunately no, because most existing analysis relies on showing

  \[ \|x^t - x^*\|^2 + \|y^t - y^*\|^2 \to 0 \]

  where \((x^*, y^*)\) are the globally optimal primal-dual pair.

How to measure the progress of the algorithm?
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Proof Ideas: A Key Step

- **Solution**: Use $L(x; y)$ as the merit function to guide the progress.

- **Challenge**: The behavior of $L(x; y)$ is difficult to characterize.
  1. Decreases after each primal update
  2. Increases after each dual update

- **Technique**: Bound the change of the dual update by that of the primal.
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  - 1. **Decreases** after each primal update
  - 2. **Increases** after each dual update

- **Technique**: Bound the change of the dual update by that of the primal
Proof Steps

- **We develop a three-step analysis framework**

- **Step 1:** Show "sufficient descent"

\[
L(\{x_k^{t+1}\}, x_0^{t+1}; y^{t+1}) - L(\{x_k^t\}, x_0^t; y^t)
\leq \sum_{k=1}^{K} \left( \frac{L_k^2}{\rho_k} - \frac{\gamma_k(\rho_k)}{2} \right) \|x_k^{t+1} - x_k^t\|^2 - \frac{\sum_{k=1}^{K} \rho_k}{2} \|x_0^{t+1} - x_0^t\|^2
\]

- **Step 2:** Show the following is "lower bounded"

\[
L(x_0^{t+1}, \{x_k^{t+1}\}; y^{t+1}) \geq -\infty
\]

- **Step 3:** Show convergence to the set of stationary solutions
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The Convergence Claim

Convergence of ADMM for Nonconvex Global Consensus

Claim: Suppose Assumption A is satisfied. Then we have

1. The linear constraint is satisfied eventually:
   \[
   \lim_{t \to \infty} \| x_k^{t+1} - x_0^{t+1} \| = 0, \quad \forall \ k
   \]

2. Any limit point of the sequence generated by Algorithm 1 is a stationary solution of the consensus problem
The Iteration Complexity Analysis

- **Need new gap function** to measure the gap to stationarity

\[ P(x^t, y^t) := \| \tilde{\nabla} L(\{x_k^t\}, x_0^t, y^t) \|^2 + \sum_{k=1}^{K} \| x_k^t - x_0^t \|^2 \]

- \( P(x, y) = 0 \iff (x, y) \) is a stationary solution

**Claim:** Suppose Assumption A is satisfied, \( \epsilon > 0 \) be some constant. Let \( T(\epsilon) \) denote an iteration index which satisfies

\[ T(\epsilon) := \min \{ t \mid P(x^t, y^t) \leq \epsilon, t \geq 0 \} \]

for some \( \epsilon > 0 \). Then there exists some constant \( C > 0 \) such that

\[ T(\epsilon) \leq \frac{C}{\epsilon}. \]
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Proximal Step and Flexible Updates

- Use **proximal gradient** to update \( x_k \) for cheap iterations

- The \( x_k \) step is replaced by

\[
x_k^{t+1} = \arg\min_{x_k} \langle \nabla g_k(x_0^{t+1}), x_k - x_0^{t+1} \rangle + \langle y_k^t, x_k - x_0^{t+1} \rangle + \frac{\rho_k + L_k}{2} \| x_k - x_0^{t+1} \|^2.
\]

- Gradient evaluated at the most recent \( x_0 \)!

- We can also use **stochastic node sampling**

- Similar convergence guarantee as Algorithm 1
Proximal Step and Flexible Updates

- Use proximal gradient to update $x_k$ for cheap iterations

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- **Similar convergence guarantee as Algorithm 1**
The Nonconvex Sharing Problem

- Our analysis also works for the well-known sharing problem [Boyd-Parikh-Chu-Peleato-Eckstein 11]

\[
\begin{align*}
\min & \quad \sum_{k=1}^{K} h_k(x_k) + g(x_0) \\
\text{s.t.} & \quad \sum_{k=1}^{K} A_k x_k = x_0, \quad x_k \in X_k, \; k = 1, \ldots, K.
\end{align*}
\]

- \( x_k \in \mathbb{R}^{N_k} \) is the variable associated with agent \( k \)

- **(K + 1)-block problem**, convergence unknown for the convex case

- **Apply our analysis to show convergence**
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Remarks

- The first analysis framework for iteration complexity of nonconvex ADMM
- A major departure from the classic analysis for convex problems
- The AL guides the convergence of the algorithm
- The $\rho_k$’s should be *large enough*, with computable lower bounds
Outline

1. Overview
2. Literature Review
3. A New Analysis Framework
   - A Toy Example
   - Nonconvex Consensus Problem
   - Algorithm and Analysis
4. Recent Advances
5. Conclusion
Recent Advances

- Many exciting recent works have been built upon our results

- New analysis, new algorithms and new connections
New Analysis

**New analysis** for weaker conditions

- **Work** [Li-Pong 14]: $h$, nonconvex, coercive; more general $A_k$; whole sequence convergence under Kurdyka-Lojasiewicz (KL) property

- **Work** [Kumar et al 16]: different update schedules

- **Work** [Bai-Scheinberg 15]: different characterization of iteration complexity

- **Works** [Jiang et al 16, Wang-Yin-Zeng 16]: both relax conditions for the $K$-agent sharing problem

- **Work** [Yang-Pong-Chen 15]: enlarges the dual stepsize by $\frac{\sqrt{5}+1}{2} \approx 1.618...$

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New Analysis

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- **Work** [Li-Pong 14]: $l$, nonconvex, coercive, $A_k$’s full row rank; whole sequence convergence under Kurdyka-Lojasiewicz (KL) property
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All based upon our analysis framework
New applications in FE, SP, ML, Comm etc.

- Risk parity portfolio selection [Bai-Scheinberg 15]
- Solving certain Hamilton-Jacobi equations and differential games [Chow-Darbon-Osher-Yin-16]
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New Connections

Connections of variants of ADMM with algorithm for convex problems

Nonconvex ADMM analysis

Generalize algorithm to nonconvex problems
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Generalize algorithm to nonconvex problems
Apply the Prox-ADMM to consensus over a general network [H. 16],

\[
\min_{\mathbf{x}} \ f(\mathbf{x}) := \sum_{i=1}^{N} f_i(x_i) \quad \text{s.t.} \quad x_i = x_j \quad \text{if} \ i, j \ \text{are neighbors}
\]
Prox-ADMM = EXTRA

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\min_x f(x) := \sum_{i=1}^{N} f_i(x_i) \quad \text{s.t.} \quad x_i = x_j \text{ if } i, j \text{ are neighbors}
\]
The resulting algorithm is equivalent to the following **primal-only** iteration

\[
x^{t+1} = x^t - \frac{1}{2\rho} D^{-1} \left( \nabla f(x^t) - \nabla f(x^{t-1}) \right) + Wx^t - \frac{1}{2} (I + W)x^{t-1}
\]

where \( D, W \) are some network-related matrices

The above iteration is precisely the **EXTRA** algorithm [Shi-Ling-Wu-Yin 14] for convex network consensus optimization

**New Claim.** EXTRA converges sublinearly for nonconvex problems
Prox-ADMM = EXTRA

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**New Claim.** EXTRA converges sublinearly for nonconvex problems
Consider the following convex finite sum problem:

$$
\min_{x \in X} f(x) := \frac{1}{N} \sum_{i=1}^{N} g_i(x),
$$

where $g_i$, $i = 1, \cdots, N$ are cost functions; $N$ is # of data points.

Many popular fast learning algorithms, like SAG [Le Roux-Schmidt-Bach 12], IAG [Blatt et al 07], SAGA [Defazio et al 14]:

1. Stochastically/deterministically pick one component function $g_i$
2. Compute its gradient
3. Update $x^{t+1}$ by using an average of the past gradients
Consider the following **convex** finite sum problem:

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Recent Advances

Prox-ADMM = SAG/IAG/SAGA

Sample one index \( i \in \{1, \cdots, N\} \), compute \( \nabla g_i(x^r) \)

\[
\begin{align*}
    z^r_i &= x^r, \quad z^r_j = z^r_{j-1}, \quad \forall \ j \neq i \\
    x^{r+1} &= x^r - \frac{1}{\beta} \sum_{j=1}^{N} \nabla g_j(z^r_{j-1}) + \frac{1}{\alpha} \left( \nabla g_i(z^r_{i-1}) - \nabla g_i(x^r) \right)
\end{align*}
\]

- Equivalent to some variants of prox-ADMM [Hajinezhad et al 16]

New Claim. SAG/IAG/SAGA converge sublinearly for nonconvex problems
Recent Advances

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z_i^r = x^r, \quad z_j^r = z_j^{r-1}, \quad \forall \ j \neq i
\]

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Recent Advances

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Sample one index $i \in \{1, \cdots, N\}$, compute $\nabla g_i(x^r)$

$z^r_i = x^r, \quad z^r_j = z^r_{j-1}, \quad \forall \ j \neq i$

$x^{r+1} = x^r - \frac{1}{\beta} \sum_{j=1}^{N} \nabla g_j(z^r_{j-1}) + \frac{1}{\alpha} \left( \nabla g_i(z^r_{i-1}) - \nabla g_i(x^r) \right)$

- Equivalent to some variants of prox-ADMM [Hajinezhad et al 16]

**New Claim.** SAG/IAG/SAGA converge sublinearly for nonconvex problems
Summary

- **Quesiton**: Whether ADMM converges for nonconvex problems?

- Yes, for a class of consensus and sharing problems, and many more

- **Key insights**
  1. The penalty parameters are required to be large enough
  2. The augmented Lagrangian measures the algorithm progress

- **Key technique**: AL as merit function, leading to a three-step analysis
Summary

This Work

New Analysis

SAG
[Defazio et al 14]

SAGA
[Le Roux et al 12]

IAG
[Blatt et al 07]

EXTRA
[Shi et al 14]

New Connections

New Applications

Signal Processing
[Defazio et al 14]

Learning
[Ames-H. 16]

Wireless
[Yatawatta 16]

Finance
[Bai-Scheinberg 15]

This Work

New Analysis

[Kumar et al 16]  
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Thank You!