Derivation of Depletion Widths
in Depletion Approximation
for an abrupt junction.

An uncompensated diode in the depletion approximation will have the following charge distribution:

Recall that we are assuming that there is no free charge (electrons and holes) in the depleted region. Thus only the exposed dopant ions remain.

We need to find the widths $x_n$, $x_p$, and $W = x_n + x_p$. Fortunately we know $N_a$, $N_d$, and

$$V_{bi} = \frac{kT}{q} \ln \left( \frac{N_a N_d}{n_i^2} \right).$$

($N_a$ and $N_d$ will be design parameters usually.)
Diagrams for Depletion Approximation

\[ p(n) \]

\[ E(x) \]

\[ V(x) \]

\[ qNd \]

\[ x_nNd = x_pN_0 \]

Shaded regions have equal area.
Thus the basic procedure is as follows:

for a p-n diode

I. Find the Electric Field in the depleted region

\[ E(x) = \frac{1}{\epsilon} \int_{-\infty}^{x} \rho(x) \, dx \]

This is Gauss's Law

II. Find Voltage, \( V(x) \) in the depleted region

\[ V(x) = -\int_{-\infty}^{x} E(x) \, dx \]

III. Set \( V(x_a) = V_{bi} \) and solve for \( x_a, x_p \) and \( W \)

So let's try it.

\[ \rho(x) = \begin{cases} 
qM_d & \text{for } 0 \leq x \leq x_a \\
-qM_a & \text{for } -x_p \leq x \leq 0 \\
0 & \text{everywhere else}
\end{cases} \]

\[ E(x) \text{ must be calculated in two regions:} \]

\[ 0 \leq x \leq x_a \text{ and } -x_p \leq x \leq 0 \]
For \(-x_p \leq x \leq 0\) (p-depletion)

\[ E(x) = \int_{-x_p}^{x} -\frac{qN_a}{e} \, dx \]
\[ = \frac{qN_a}{e} x \Big|_{-x_p}^{x} = -\frac{qN_a}{e} x - \frac{qN_a}{e}(-x_p) \]
\[ E(x) = -\frac{qN_a}{e}(x + x_p) \quad \text{for} \quad -x_p \leq x \leq 0 \]

For \(0 \leq x \leq x_n\) (n-depletion)

\[ E(x) = \int_{-x_p}^{0} -\frac{qN_a}{e} \, dx + \int_{0}^{x} \frac{qN_d}{e} \, dx \]
\[ E(x) = -\frac{qN_a}{e} x_p + \frac{qN_d}{e} x \Big|_{0}^{x} \]
\[ E(x) = -\frac{qN_a}{e} x_p + \frac{qN_d}{e} x \quad \text{for} \quad 0 \leq x \leq x_n \]

At this point, we should simplify the above expression. To do this, let's develop an expression for the charge on both sides of the junction. Since the diode as a whole is charge neutral, the charge on each side of the junction must be equal and opposite.
\[-q \text{Na}_x \times_0 A = q \text{Nd}_x x A \]

Thus we can rewrite

\[ E(x) = -\frac{q \text{Na}_x x_0}{\varepsilon} + \frac{q \text{Nd}_x x}{\varepsilon} \quad \text{for} \quad 0 \leq x \leq x_a \]

dividing the equation by \(\varepsilon\)

\[ E(x) = -\frac{q \text{Nd}_x x_0}{\varepsilon} + \frac{q \text{Nd}_x x}{\varepsilon} \]

\[ E(x) = -\frac{q \text{Nd}_x}{\varepsilon} (x_a - x) \quad \text{for} \quad 0 \leq x \leq x_a \]

\[ E(x) = \begin{cases} -\frac{q \text{Na}_x}{\varepsilon} (x_0 + x) & \text{for} \quad -x_0 \leq x \leq 0 \\ -\frac{q \text{Nd}_x}{\varepsilon} (x_a - x) & \text{for} \quad 0 \leq x \leq x_a \end{cases} \]

\[ V(x) = -\int_{-\infty}^{x} E(x) \, dx \]

First take region \(-x_0 \leq x \leq 0 \) (\(\rho\)-depletion)

\[ V(x) = -\int_{-x_0}^{x} -\frac{q \text{Na}_x}{\varepsilon} (x_0 + x) \, dx \quad \text{for} \quad -x_0 \leq x \leq 0 \]

\[ = \left[ -\frac{q \text{Na}_x}{\varepsilon} x_0 x + \frac{q \text{Na}_x x^2}{2 \varepsilon} \right]_{-x_0}^{x} \]
\[ V(x) = \frac{qN_0}{e} x_0 + \frac{qN_0}{e} x_0^2 + \frac{qN_0}{2e} x^2 - \frac{qN_0}{2e} x_0^2 \]

\[ V(x) = \frac{qN_0}{e} (x_0 x + x_0^2 + \frac{1}{2} x^2 - \frac{1}{2} x_0^2) \]

\[ V(x) = \frac{qN_0}{2e} (x^2 + 2x_0 x + x_0^2) \]

\[ V(x) = \frac{qN_0}{2e} (x + x_0^2) \quad \text{for} \quad -x_0 \leq x \leq 0 \]

**Now take region** \( 0 \leq x \leq x_0 \)

\[ V(x) = -\int_{-\infty}^{x} E(x) \text{d}x \]

\[ = -\left[ \int_{-\infty}^{0} E(x) \text{d}x + \int_{0}^{x} E(x) \text{d}x \right] \quad \text{for} \quad 0 \leq x \leq x_0 \]

\[ = \frac{qN_0}{2e} x_0^2 + \int_{0}^{x} \frac{qN_0}{e} (x_0 - x) \text{d}x \]

\[ = \frac{qN_0}{2e} x_0^2 + \left( \frac{qN_0}{e} x_0 x - \frac{qN_0}{e} \frac{x^2}{2} \right) \bigg|_{0}^{x} \]

\[ V(x) = \frac{qN_0}{2e} x_0^2 + \frac{qN_0}{e} x_0 x - \frac{qN_0}{e} \frac{x^2}{2} \]

At \( x = x_0 \), \( V(x_0) = V_i \)

\[ \therefore V_i = \frac{qN_0}{2e} x_0^2 + \frac{qN_0}{2e} x_0^2 \]

\[ \frac{qN_0}{2e} x_0^2 = V_i - \frac{qN_0}{2e} x_0^2 \quad (\ast) \]
Now use eqn. (*) to rewrite $V(x)$

$$V(x) = V_{bi} - \frac{qN_d}{2e} x_n^2 + \frac{qN_a}{2e} 2x_n x - \frac{qN_d}{2e} x^2$$

$$V(x) = V_{bi} - \frac{qN_d}{2e} (x+x_n)^2 \text{ for } 0 \leq x \leq x_n$$

$$V(x_n) = V_{bi}$$

$$\frac{qN_d}{2e} x_n^2 + \frac{qN_a}{2e} x_n^2 = V_{bi}$$

Also

$$x_n N_d = x_p N_a$$

$$x_p = x_n \frac{N_d}{N_a}$$

$$\frac{qN_d}{2e} x_n^2 + \frac{qN_a}{2e} \frac{N_d^2}{N_a} x_n^2 = V_{bi}$$

$$x_n^2 = \frac{2e}{q} V_{bi} \frac{1}{\left[ \frac{N_d}{N_a} + 1 \right]}$$

$$x_n = \left[ \frac{2e}{q} V_{bi} \frac{N_a}{N_d (N_d + N_a)} \right]^{1/2}$$

Similarly for $x_p$

$$x_p = \left[ \frac{2e}{q} V_{bi} \frac{N_d}{N_a (N_a + N_d)} \right]^{1/2}$$
And the total width of the depletion region is:

\[ W = x_n + x_p \]

\[ W = \left[ \frac{2eV_d}{q} \cdot \frac{N_A + N_D}{N_A N_D} \right]^{1/2} \]

From \( x_n \) and \( x_p \) we can see that the depletion region extends further into a lightly doped region rather than a more heavily doped one. This is due to the necessity to have an equal amount of depletion charge on both sides of the junction.
Example 1:

For the following n-p junction, draw the band diagram and find $V_0$, $x_n$, $x_p$, $W$. Also draw $P(x)$, $E(x)$, and $V(x)$ and find $\phi_n$ and $\phi_p$.

\[
\begin{array}{c|c}
N & P \\
\hline
N_d = 4 \times 10^{16} & N_o = 8 \times 10^{16}
\end{array}
\]

\[q \phi_n = \text{E}_f - \text{E}_i = -kT \ln \frac{n_i}{n_n} = 0.45 \text{eV}\]

\[q \phi_p = \text{E}_f - \text{E}_i = kT \ln \frac{N_o}{N_p} = -0.41 \text{eV}\]

\[V_{0i} = |\phi_n| + |\phi_p| = 0.86 \text{V}\]

\[x_p = \sqrt{\frac{2eV_i}{q} \frac{N_d}{N_d(N_d+N_o)}} = 0.11 \mu\text{m}\]

\[x_n = \sqrt{\frac{2eV_i}{q} \frac{N_o}{N_d(N_d+N_o)}} = 0.022 \mu\text{m}\]

\[W = x_p + x_n = 0.132 \mu\text{m}\]
\[ \varepsilon(x) = +\frac{q N_e}{\varepsilon} x_0 = +\frac{q N_e x_1}{\varepsilon} \]

\[ \varepsilon(x) = \int \frac{\rho(x)}{\varepsilon} \, dx \]

\[ V(x) = -\int \varepsilon(x) \, dx \]
Example 2

Sketch $p(x)$, $E(x)$, $V(x)$ for the following device in equilibrium.

\[ E(x) = \int \frac{p(x)}{\varepsilon} \, dx \]

\[ V(x) = -\int E(x) \, dx \]