to any tangent hyperplane to $U(x)$ at $V_2(x) = B$ is bounded away from 0 and ∞.

To conclude, we have established the following finite algorithm for computing optimal strategies for the constrained problem.

Algorithm 3:

1. Check for the cases in Theorems 2 and 3.
2. If they do not hold, compute bounds for b using Lemma 6 and Algorithm 2.
3. Compute N using [4, Algorithm 3.7].
4. Find any (i, j)-lexicographic-optimal stationary strategy σ and compute $V_k(y, \sigma),\ k = 1, 2$.
5. Use a finite horizon algorithm to compute an optimal strategy.

Remark: Consider the following two-dimensional constrained optimization problem:

$$\begin{aligned}\text{maximize} \quad & \sum_{k=1}^{2} a_k V_k(x, \pi)\\ \text{subject to} \quad & \sum_{k=1}^{2} c_k V_k(x, \pi) \geq B.\end{aligned}$$

Then Theorems 2–4 continue to hold, except that $N(b)$ is not monotone. The argument of Theorem 5 continues to hold as well. However, the computation of a bound on N (or, equivalently, on b) is more complicated.

Remark: Some of the ideas herein extend to the higher dimensional problem

$$\begin{aligned}\text{maximize} \quad & V_1(x, \pi)\\ \text{subject to} \quad & V_k(x, \pi) \geq B_k, \quad k = 2, \ldots, K\end{aligned}$$

where each V_k is a standard discounted criterion with discount factor β_k. For example, if

$$\max_{\pi} V_k(x, \pi) = B_k$$

for some $k \geq 2$, then we can restrict to actions which are conserving for V_k, and consider the problem with $K - 2$ constraints. Due to the higher dimentionality, however, the geometry is less transparent and there are more specific cases to deal with.

ACKNOWLEDGMENT

This work was performed in part while A. Shwartz was on sabbatical leave. The hospitality and support of the Mathematical Center, Bell Laboratories, Murray Hill, NJ, and of the Department of Management Science and Information Systems, Graduate School of Business, Rutgers University, New Brunswick, NJ, is gratefully acknowledged.

REFERENCES

[1] E. Altman, E. Feinberg, and A. Shwartz, "Weighted discounted stochastic games with perfect information," in *Proc. Seventh Int. Symp. Dynamic Games and Applications*, J. A. Filar, V. Gaitsgory, and F. Imado, Eds., Kanagawa, Japan, Dec. 16–18, 1996, vol. 1, pp. 18–31.

[2] E. Feinberg, "Controlled Markov processes with arbitrary numerical criteria," *SIAM Theory Prob. Appl.*, vol. 27, pp. 486–503, 1982.

[3] E. Feinberg, "Constrained discounted Markov decision processes and Hamiltonian cycles," in *Proc. 36th IEEE Conf. Decision and Control*, San Diego, CA, Dec. 10–12, 1997, vol. 3, pp. 2821–2826.

[4] E. Feinberg and A. Shwartz, "Markov decision models with weighted discounted criteria," *Math. Oper. Res.*, vol. 19, pp. 152–168, 1994.

[5] ——, "Constrained Markov decision models with weighted discounted rewards," *Math. Oper. Res.*, vol. 20, pp. 302–320, 1995.

[6] E. Feinberg and A. Shwartz, "Constrained discounted dynamic programming," *Math. Oper. Res.*, vol. 21, pp. 922–945, 1996.

[7] E. Fernandez-Gaucherand, M. K. Ghosh, and S. I. Marcus, "Controlled Markov processes on the infinite planning horizon: Weighted and overtaking cost criteria," *ZOR-Methods and Models in Operations Research*, vol. 39, pp. 131–155, 1994.

[8] J. Filar and O. J. Vrieze, "Weighted reward criteria in competitive Markov decision processes," *ZOR-Methods and Models of Operations Research*, vol. 36, pp. 343–358, 1992.

[9] M. K. Ghosh and S. I. Marcus, "Infinite horizon controlled diffusion problems with some nonstandard criteria," *J. Mathematical Systems, Estimation and Control*, vol. 1, pp. 45–69, 1991.

[10] R. L. Keeney and H. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. New York: Wiley, 1976.

[11] D. Krass, J. Filar, and S. S. Sinha, "A weighted Markov decision process," *Oper. Res.*, vol. 40, pp. 1180–1187, 1992.

[12] D. Krass, "Contributions to the theory and applications of Markov decision processes," Ph.D. thesis, The Johns Hopkins Univ., Baltimore, MD, 1989.

The Interpolation Problem with a Degree Constraint

Tryphon T. Georgiou

***Abstract*— In [6]–[8] it was shown that there is a correspondence between *nonnegative* (hermitian) trigonometric polynomials of degree $\leq n$ and solutions to the standard Nevanlinna–Pick–Carathéodory interpolation problem with $n + 1$ constraints, which are rational and also of degree $\leq n$. It was conjectured that the correspondence under suitable normalization is bijective and thereby, that it results in a complete parametrization of rational solutions of degree $\leq n$. The conjecture was proven in an insightful work by Byrnes *etal*. [1], along with a detailed study of this parametrization. However, the result in [1] was shown under a slightly restrictive assumption that the trigonometric polynomials are *positive* and accordingly, the corresponding solutions have *positive* real part. The purpose of the present note is to extend the result to the case of *nonnegative* trigonometric polynomials as well. We present the arguments in the context of the general Nevanlinna–Pick–Caratheodory–Fejér interpolation.**

NOTATION

$\mathbf{C}$	Complex numbers.
$\mathcal{D}$	Open unit disc $\mathcal{D}$:= open unit disc $= \{z \in \mathbf{C} : \lvert z\rvert < 1\}$.
$\mathcal{X}^c$, $\mathcal{X}^o$, $\partial\mathcal{X}$	Closure, open interior, and boundary of a set $\mathcal{X}$ resp.
$\mathcal{H}(\mathcal{D})$	Set of functions holomorphic in $\mathcal{D}$.
$\mathcal{C}$	Carathéodory class $\{f(z) \in \mathcal{H}(\mathcal{D}) : \Re\{f(z)\} \geq 0$ for all $z \in \mathcal{D}\}$.

Manuscript received July 15, 1997. Recommended by Associate Editor, A. Varga. This work was supported in part by AFOSR under Grant AF-F49620-96-1-0094 and NSF under Grant NSF-ECS-9016050.

The author is with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455 USA (e-mail: georgiou@ece.umn.edu).

Publisher Item Identifier S 0018-9286(99)01317-3.

$\mathcal{S}$	Schur class $\{f(z) \in \mathcal{H}(\mathcal{D}) : \|f(z)\| \leq 1$ for all $z \in \mathcal{D}\}$.
z^*	Complex conjugate of $z \in \mathbf{C}$.
$\mathcal{L}_2$	Space of squarely integrable functions on $\partial\mathcal{D}$.
$\mathcal{H}_2$	Hardy space of $\mathcal{L}_2$-functions that have analytic continuation in $\mathcal{D}$.

If $a(z) = a_0 + a_1 z + \cdots \in \mathcal{H}(\mathcal{D})$, then $a(z)_* := a_0^* + a_1^* z^{-1} + \cdots$.

I. Introduction and Results

Consider two sets of $n+1$ points in $\mathcal{D}$ and $\mathbf{C}$ respectively,

$$\mathcal{Z} := \{z_k : k = 0, 1, \ldots, n\} \quad \text{and} \quad \mathcal{W} := \{w_k : k = 0, 1, \ldots, n\}.$$

The standard Nevanlinna–Pick–Carathéodory–Fejér interpolation problem is stated as follows.

Problem $\mathbf{NP}(\mathcal{Z}, \mathcal{W})$*:* Determine whether a function $f(z) \in \mathcal{C}$ exists that satisfies the interpolation conditions

$$f(z_k) = w_k, \qquad \text{for } k = 0, 1, \ldots, n. \tag{1}$$

If such a function exists, characterize all such solutions.

If points in $\mathcal{Z}$ are not distinct, e.g., $z_k = z_1$ $(k = 1, \ldots, j)$, then the interpolation conditions involve the derivatives of $f(z)$ and are rephrased by requiring that $\frac{1}{(k-1)!} f^{(k-1)}(z_1) = w_k$ $(k = 1, \ldots, j)$ accordingly, where $f^{(k-1)}(z)$ denotes the $(k-1)$th derivative. In particular, if all interpolating points z_k $(k = 1, \ldots, n)$ coincide, then it is the value of $f(z)$ along with its first n derivatives that are specified at that point (usually the origin). This is known as the Carathéodory problem.

In general, it is known that a solution for $\mathbf{NP}(\mathcal{Z}, \mathcal{W})$ exists if and only if an associated *Pick matrix* is positive semidefinite. In case all z_k's are distinct the Pick matrix is given by

$$P = \left[\frac{w_k + w_\ell^*}{1 - z_k z_\ell^*}\right]_{k,\ell=0}^{n}$$

while, in the context of the Carathéodory problem

$$P = \begin{bmatrix} w_0 + w_0^* & w_1^* & \cdots & w_n^* \\ w_1 & w_0 + w_0^* & \cdots & w_{n-1}^* \\ \vdots & \vdots & & \vdots \\ w_n & w_{n-1} & \cdots & w_0 + w_0^* \end{bmatrix}$$

which in this case is a *Toeplitz matrix*. In general, P is the real part of the "compressed" operator $\mathbf{\Pi}_\mathcal{K} f_0(z)|_\mathcal{K}$; see Remark 1 below. If the Pick matrix is singular, the solution is unique, rational and of degree $\leq n$, while if $P > 0$ *all* solutions can be described via a linear fraction transformation on arbitrary elements in $\mathcal{C}$. An exposition of the classical mathematical theory is presented in [11], and an independent approach in the context of passive circuits in [12].

$\mathbf{NP}(\mathcal{Z}, \mathcal{W})$ and related interpolation problems (associated with the names of O. Toeplitz, C. Carathéodory, L. Fejér, I. Schur, N. I. Akhiezer, M. G. Kreĭn, V. P. Potapov, and others) have been studied since the beginning of the century, and have spawned a body of beautiful mathematics culminating in some deep results in functional analysis (by B. Sz.-Nagy, C. Foiaş, D. Sarason, J. Ball, J. W. Helton, and others). Moreover, such problems have found applications in a variety of engineering fields (robust control, $\mathcal{H}_\infty$-control, approximation, circuit theory, signal processing). In engineering applications it is often desirable for the interpolating function $f(z)$ to be rational and of small degree, cf. [12]. (Here, the *degree of a rational function* is defined to be the largest of the degrees of denominator and numerator since the functions are taken to be analytic inside the unit disc.) The degree of the transfer function relates to the dimension of a controller, a filter, or a model of a stochastic process, depending on the context, cf. [12], [8], [1] and the references therein.

Motivated by the need to characterize solutions of a given degree, it was recognized in [6] that the set of solutions of degree $\leq n$ may have a nice structure. Thus, $\mathbf{NP}(\mathcal{Z}, \mathcal{W})$ was looked at as a problem of solving a set of nonlinear equations allowing only for degree $\leq n$ interpolants. The approach was topological making use of degree theory. The main result in [7] (cf. [6], [8]), given below, describes rational solutions of degree $\leq n$.

Theorem 1 [7, Theorem 5.3]: Consider $\mathbf{NP}(\mathcal{Z}, \mathcal{W})$ and assume that $P > 0$. Given any polynomial $\eta(z) \not\equiv 0$ of degree $\leq n$, having all its roots in $\{z : |z| \geq 1\}$, there exists a pair of polynomials $(\pi(z), \chi(z))$ of degree $\leq n$ such that $f(z) = \pi(z)/\chi(z)$ is in $\mathcal{C}$, $f(z)$ satisfies the interpolation conditions (1), and moreover

$$\pi(z)\chi(z)_* + \chi(z)\pi(z)_* = \kappa^2 \eta(z)\eta(z)_* \tag{2}$$

for some $\kappa > 0$.

The proof in [7] was presented for the case where $\mathcal{Z}$ is a set of discrete points. However, with the obvious notational adjustments the proof carries over to the general case where some of the elements in $\mathcal{Z}$ may coincide, and will not be repeated here. In particular, the result for the Carathéodory problem where all points in $\mathcal{Z}$ coincide was documented in [8, Theorem 3.1], and for matrix interpolation in [6, Theorem 9.4].

Thus, to recap, the theorem states that for any (hermitian) *nonnegative* trigonometric polynomial $d(z, z^{-1}) = \eta(z)\eta(z)_*$ of degree $\leq n$, there is a corresponding rational interpolating function $f(z) \in \mathcal{C}$ also of degree $\leq n$. Conversely, it is obvious that to any rational interpolating function $f(z) = \pi(z)/\chi(z) \in \mathcal{C}$ we can associate a nonnegative trigonometric polynomial $d(z, z^{-1}) = \pi(z)\chi(z)_* + \chi(z)\pi(z)_*$, or equivalently, the "stable" spectral factor $\eta(z)$, of similar degree.

In general, the correspondence $\eta(z) \to f(z)$ may not be one-to-one due to possible cancellations between π and χ; e.g., for $n = 1$, $w_0 = w_1 = 1$ and arbitrary "stable" $\eta(z)$ of degree ≤ 1 we get $\pi(z) = \chi(z) = \eta(z)$ and $f(z) \equiv 1$. In fact, such a cancellation between $\pi(z)$ and $\chi(z)$ occurs precisely when there exists a solution of degree strictly less than n. Interestingly, the correspondence between $\eta(z)$ (suitably normalized, e.g., by $\eta(0) = 1$) and the "graph symbol" $(\pi(z), \chi(z))$ of $f(z)$ is actually one-to-one (provided $\pi(z)$ and $\chi(z)$ are chosen to be "stable"). This fact was conjectured in [6] and remained open until the recent work by Byrnes, Lindquist, Gusev, and Matveev [1]. In [1, Corollary 2.3] the authors proved the conjecture (in the context of the Carathéodory/covariance realization problem) under the slightly restrictive assumption that $\eta(z)$ has no roots on $\{z : |z| = 1\}$, or equivalently, under the assumption that $d(z, z^{-1})$ is a positive trigonometric polynomial for $z = e^{i\theta}$ $(\theta \in [-\pi, \pi])$. It should be noted that [1] focused on other important questions about the map, and in particular, showed that it is an analytic diffeomorphism (on polynomials η with roots in $\{z : |z| > 1\}$). In subsequent work, Byrnes, Landau, and Lindquist [3], have given an alternative proof of the correspondence being bijective under the same assumption that $d(z, z^{-1})$ is a positive trigonometric polynomial. The case where $d(z, z^{-1})$ is only nonnegative was left as an open problem (see [3, Sec. 5]).

The purpose of the present note is to show that the correspondence is bijective even for the case where $d(z, z^{-1})$ is nonnegative (or, equivalently, when $\eta(z)$ is allowed to have roots on the boundary $\{z : |z| = 1\}$). The first part of the argument given below proves that any two pairs $(\pi_j(z), \chi_j(z))$ $(j = 1, 2)$ satisfying the conditions of Theorem 1, with $\chi_j(z)$ devoid of roots on the unit circle, must be identical. The particular argument is an adaptation of a similar argument in [3] (cf. [1, Lemma 4.5]) given in the context of the

Carathéodory problem. It is used here because it easily generalizes to the case of interpolation with complex values—a step which is necessary for the second part of the proof. The second part makes use of a bijective transformation of a pair $(\pi_j(z), \chi_j(z))$ (possibly having roots on the boundary), which preserves the corresponding $\eta(z)$, avoids the possibility of poles on the boundary, and leads to interpolating values for the transformed pair which depend only on the initial data. Then, invoking the first part of the argument, any two such functions $f_j = \pi_j/\chi_j$ must be identical. The precise statement is given below as Theorem 2 (cf. [1, Corollary 2.3]).

Theorem 2: For any polynomial $\eta(z) \not\equiv 0$, normalized by $\eta(0) = 1$, with degree $\leq n$ and roots in $\{z : |z| \geq 1\}$, there is a unique pair of polynomials $(\pi(z), \chi(z))$ such that $\pi(z) + \chi(z)$ has all its roots in $|z| \geq 1$, and $\pi(z)$, $\chi(z)$ satisfy the conditions of Theorem 1; that is, they have degree $\leq n$, the function $f(z) = \pi(z)/\chi(z)$ is in $\mathcal{C}$ and satisfies the interpolation conditions (1), and

$$\pi(z)\chi(z)_* + \chi(z)\pi(z)_* = \kappa^2\eta(z)\eta(z)_* \tag{3}$$

for some $\kappa > 0$. Furthermore, any root of $\pi(z) + \chi(z)$ on $|z| = 1$ is common to all three polynomials $\pi(z)$, $\chi(z)$ and $\eta(z)$, in which case $f(z) = \pi(z)/\chi(z)$ is an interpolating function of degree $< n$.

It should be noted that individually, π and/or χ, may have distinct roots on $\partial\mathcal{D}$. Hence, $\pi(z)/\chi(z)$ is in $\mathcal{C}$, though not necessarily a strictly positive-real function. The interest for extending the parametrization to $\mathcal{C}$-functions stems from the fact that it is precisely $\mathcal{C}$-membership which characterizes impedance of passive systems (see [12]), power spectra (see [8]), etc. Before we proceed with the proof, we introduce notation and key relevant facts. An alternative proof, which applies to the special case where $\eta(z)$ has no root on $\partial\mathcal{D}$ is presented in [5] following [4].

II. The Coinvariant Subspace $\mathcal{K}$ and Lagrange Interpolation

Let $\mathcal{K} = \mathcal{H}_2 \ominus B(z)\mathcal{H}_2$, where $B(z)$ is the Blaschke product

$$\prod_{k=0}^{n} \frac{z - z_k}{1 - z_k^* z} \cdot \frac{|z_k|}{z_k}$$

and $\frac{|z_k|}{z_k}$ is replaced by 1 when $z_k = 0$. Clearly, $\mathcal{K}$ is an $(n+1)$-dimensional space. In case all the z_k's are distinct, $\mathcal{K}$ admits a basis $\mathcal{B} = \{g_k(z), k = 0, 1, \ldots, n\}$ such that for any function $f(z) \in \mathcal{H}(\mathcal{D})$ which is continuous on the boundary

$$\begin{aligned}\langle f(z), g_k(z)\rangle &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})g_k(e^{i\theta})^*\, d\theta \\ &= f(z_k).\end{aligned}$$

(Here, $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathcal{L}_2(\partial D)$.) In case any of the z_k's coincide, e.g., if $z_k = z_{k+1} = \cdots = z_{k+\ell}$, then $\mathcal{K}$ admits a basis such that $\langle f(z), g_{k+m}(z)\rangle = f^{(m)}(z_k)$ for $m = 0, 1, \ldots, \ell - 1$, where $f^{(m)}(z)$ denotes the mth derivative of $f(z)$. For instance, in the two extreme cases where (a) all z_k's are distinct or, (b) all z_k's coincide, we have

$$\mathcal{B} = \left\{g_k(z) = \frac{1}{1 - z_k^* z},\ k = 0, 1, \ldots, n\right\}$$

and

$$\mathcal{B} = \left\{g_k(z) = \frac{z^k}{k!(1 - z_0^* z)^{k+1}},\ k = 0, 1, \ldots, n\right\}$$

respectively. (The space $\mathcal{K}$ is referred to as *coinvariant* because it is invariant under the adjoint of the natural shift operator z, a fact which is not explicitly used herein.)

Any element $q(z) \in \mathcal{K}$ is simply a rational function $q(z) = \chi(z)/r(z)$ where $\chi(z)$ is a polynomial of degree $\leq n$ and

$$r(z) := \prod_{k=0}^{n}(1 - z_k^* z).$$

Below, we choose to represent any rational function $f(z)$ of degree $\leq n$ as a fraction of two elements $p, q \in \mathcal{K}$, i.e.,

$$f(z) = \frac{p(z)}{q(z)}, \qquad \text{with } p(z) = \frac{\pi(z)}{r(z)}, \quad q(z) = \frac{\chi(z)}{r(z)} \in \mathcal{K}$$

where $\pi(z), \chi(z)$ denote suitable polynomials of degree $\leq n$.

Now, let $\mathbf{\Pi}_{\mathcal{K}}$ denote the orthogonal projection onto $\mathcal{K}$, and let $f_0(z)$ be a polynomial solution to the Lagrange interpolation problem with data $(\mathcal{Z}, \mathcal{W})$. Thus $f_0(z)$ is analytic on the whole plane. For any $q(z) \in \mathcal{K}$ devoid of roots in $\mathcal{D}$, define

$$p(z) = \mathbf{T}q(z) := \mathbf{\Pi}_{\mathcal{K}} f_0(z)q(z).$$

It turns out that $f(z) = p(z)/q(z)$ satisfies the interpolation conditions ([7, Lemma 6.1]) since

$$\begin{aligned}p(z_k) &= \langle p(z), g_k(z)\rangle \\ &= \langle \mathbf{\Pi}_{\mathcal{K}} f_0(z)q(z), g_k(z)\rangle \\ &= \langle f_0(z)q(z), g_k(z)\rangle = f_0(z_k)q(z_k).\end{aligned}$$

The same is true in case of derivative interpolation, e.g., if $\langle f(z), g_{k+1}(z)\rangle = f^{(1)}(z_k)$ then $p^{(1)}(z_k) = \langle p(z), g_{k+1}(z)\rangle = \cdots = f_0^{(1)}(z_k)q(z_k) + f_0(z_k)q^{(1)}(z_k)$ from which, using $p(z_k) = f_0(z_k)q(z_k)$, we can show that $f_0^{(1)}(z_k) = (p(z)/q(z))^{(1)}|_{z_k}$. Cases with higher multiplicity proceed similarly.

In general, a function $f(z) = p(z)/q(z)$ constructed as above is only a Lagrange interpolant since it may not have the required analytic properties.

Remark 1: It is interesting to note ([7, Proposition 6.2], cf. [10]) that $\mathbf{T}$ depends only on the interpolation data, and not on the particular $f_0(z)$ chosen. Moreover, the Pick matrix is precisely the real part of $\mathbf{T}$.

III. Positive-Realness Conditions

A function $f(z) \in \mathcal{C}$ is analytic in $\mathcal{D}$ but may have singularities on the unit circle (e.g., $f(z) = \frac{1-z}{1+z}$). However, in case $f(z) \in \mathcal{C}$ is rational, then $(1 - f(z))/(1 + f(z))$ is bounded by 1 in $\mathcal{D}$ and therefore can have no singularities (poles) on the boundary. In fact (e.g., see [12]), a rational function $f(z) = \frac{p(z)}{q(z)}$, with p and q having no common root, is in $\mathcal{C}$ if and only if $q(z) + p(z)$ has no roots in $\mathcal{D}^c$ and

$$\begin{aligned}d_{p,q}(z, z^{-1}) &:= p(z)q(z)_* + q(z)p(z)_* \\ &= \frac{((q+p)(q+p)_* - (q-p)(q-p)_*)}{4} \\ &\geq 0 \quad \text{for } z = e^{i\theta}, \quad \theta \in [-\pi, \pi].\end{aligned}$$

In case a rational function $f(z) = p(z)/q(z) = \pi(z)/\chi(z) \in \mathcal{C}$ has singularities on the unit circle, it is easy to see from the fact that $\pi(z)\chi(z)_* + \chi(z)\pi(z)_*$ is a nonnegative trigonometric polynomial factorable as $\kappa^2\eta(z)\eta(z)_*$ (η being a polynomial in z), that the roots of $\chi(z)$ on the circle are also roots of $\eta(z)$. Conversely, if $\eta(z)$ has no roots on the circle, neither does $\chi(z)$ and therefore $f(z)$ is continuous on the boundary.

Proof of Theorem 2: The existence of a pair $(\pi(z), \chi(z))$ as claimed, follows from Theorem 1. We note that the proof of Theorem 1 given in [7] (also [6], [8]) establishes that there exists such a pair for which $\pi(z)+\chi(z)$ has no roots in $|z|<1$. However, roots on the boundary are possible. If this is the case, then from

$$\begin{aligned}\pi\chi_* + \pi_*\chi &= \frac{1}{2}((\chi+\pi)(\chi+\pi)_* - (\chi-\pi)(\chi-\pi)_*)\\ &\geq 0 \quad \text{for } z=e^{i\theta}, \quad \theta\in[-\pi,\pi]\end{aligned}$$

it follows that $\chi-\pi$ shares the same roots as $\chi+\pi$ on $|z|=1$. Therefore these are roots of π, χ and η as well. Hence, when the common factors are cancelled between π and χ, $f(z)=\pi(z)/\chi(z)$ is a $\mathcal{C}$-function of degree strictly less than n that satisfies the interpolation conditions. This proves the last part of the statement of Theorem 2.

We now proceed to prove uniqueness of the pair $(\pi(z), \chi(z))$. The proof has two parts. The first part deals with the case where $f(z)$ is continuous on the boundary. The argument we use here is an adaptation of a similar argument in [3] given in the context of the Carathéodory problem (also, cf. [1, Lemma 4.5]). This argument generalizes easily to the case of interpolation with complex values—a step needed for the second part of the proof. In the second part of the proof we transform pairs (π, χ) in such a way so as to remove singularities on the boundary, and then use the conclusion of first part of the proof.

From now on we assume that

$$z_0 = 0 \quad \text{and} \quad w_0 = 1. \tag{4}$$

This is without loss of generality since a conformal mapping can bring z_0 to the origin without altering the properties of the functions considered, and real-scaling and addition of an imaginary number to the data $\mathcal{W}$ translates the problem into an equivalent one satisfying (4).

Consider a rational solution $f(z)=\pi(z)/\chi(z)\in\mathcal{C}$ of degree $\leq n$ which satisfies the interpolation conditions and is *continuous* on the boundary. In this case

$$\begin{aligned}\langle f(z), g_k(z)\rangle &= f(z_k) = w_k\\ \langle f(z)_*, g_k(z)\rangle &= f(0)^* = 1\\ \langle f(z)_*, g_k(z)_*\rangle &= f(z_k)^* = w_k^*\\ \langle f(z), g_k(z)_*\rangle &= f(0) = 1\end{aligned}$$

assuming that the z_k's are distinct. If some of the z_k's are not distinct, then $\langle f(z)_*, g_k(z)\rangle$ is either one or zero (e.g., if $z_1=0$, then $\langle f(z)_*, g_1(z)\rangle = 0$, etc.) and similarly for $\langle f(z), g_k(z)_*\rangle$. In any case, the values depend on the interpolating conditions and not on the particular function chosen.

Now let $f_j(z)=\pi_j(z)/\chi_j(z)$ $(j=1,2)$ be two such interpolating functions, where π_j, χ_j are polynomials of degree $\leq n$, and

$$\pi_j(z)\chi_j(z)_* + \chi_j(z)\pi_j(z)_* = \kappa_j^2\eta(z)\eta(z)_*.$$

Define

$$\begin{aligned}p_j(z) &:= \frac{1}{\kappa_j}\pi_j(z)/r(z)\in\mathcal{K}\\ q_j(z) &:= \frac{1}{\kappa_j}\chi_j(z)/r(z)\in\mathcal{K}\\ u(z) &:= \eta(z)/r(z)\in\mathcal{K}\end{aligned}$$

and note that, since the two functions interpolate the same values

$$\langle f_1(z) - f_2(z), \cdot\rangle = 0$$

on

$$\mathcal{S} := \mathrm{span}\{g_k(z), g_{k*}(z) : k=0,1,\ldots,n\}.$$

Next, note that $q_j(z)q_{j*}(z)\in\mathcal{S}$ $(j=1,2)$. Hence

$$0 = \langle f_1(z) - f_2(z), q_2(z)q_{2*}(z) - q_1(z)q_{1*}(z)\rangle.$$

It follows that

$$\begin{aligned}0 &= \langle \mathcal{R}(f_1(z)-f_2(z)), q_2(z)q_{2*}(z) - q_1(z)q_{1*}(z)\rangle\\ &= \Big\langle u(z)u_*(z)\Big(\frac{1}{q_1(z)q_{1*}(z)} - \frac{1}{q_2(z)q_{2*}(z)}\Big),\\ &\qquad q_2(z)q_{2*}(z) - q_1(z)q_{1*}(z)\Big\rangle\\ &= \Big\langle \frac{u(z)u_*(z)}{q_1(z)q_{1*}(z)q_2(z)q_{2*}(z)}, |q_2(z)q_{2*}(z) - q_1(z)q_{1*}(z)|^2\Big\rangle.\end{aligned}$$

Since the first entry in $\langle\cdot,\cdot\rangle$ above is positive a.e. on $[-\pi,\pi]$, it follows that

$$(|q_2(e^{i\theta})|^2 - |q_1(e^{i\theta})|^2)^2 = 0, \qquad \text{for all values } \theta\in[-\pi,\pi].$$

Hence $q_1(z)=q_2(z)$. Since f_1, f_2 assume the same values at the roots of $B(z)$, $f_1(z)-f_2(z)$ has $B(z)$ as a factor. Therefore, $p_1(z)-p_2(z)\in B(z)\mathcal{H}_2$. But p_1, p_2 are both in $\mathcal{K}$. Hence $p_1(z)=p_2(z)$, and therefore $f_1(z)=f_2(z)$ as well.

Now consider the case where $f_j(z)$ $(j=1,2)$ may be singular on the unit circle, and let

$$h_j(z) = \frac{q_j(z)-p_j(z)}{q_j(z)+p_j(z)} =: \frac{a_j(z)}{b_j(z)}.$$

(We note that $h_j\in\mathcal{S}$.) Choose $\phi\in[-\pi,\pi]$ such that

$$e^{i\phi}a_j(z)+b_j(z) \quad \text{and} \quad \eta(z) \text{ have no common root on } \partial\mathcal{D}$$

for both $j=1,2$. This can always be done provided $p_j(z), q_j(z)$, and consequently $a_j(z), b_j(z)$ have no common roots for each j at the same point on $\partial\mathcal{D}$. (If there is a common root on the circle, then this is a root of both numerator and denominator for f or h and a root of η as well, and thus does not affect the rest of the argument.) Next consider

$$\hat{f}_j(z) := \frac{1-e^{i\phi}a_j(z)/b_j(z)}{1+e^{i\phi}a_j(z)/b_j(z)}, \qquad \text{for } j=1,2.$$

These are clearly in $\mathcal{C}$ [as can be seen by noting (a) that $\sigma : f \longrightarrow \frac{1-f}{1+f}$ maps $\mathcal{C}$ to $\mathcal{S}$ and $\mathcal{S}$ back to $\mathcal{C}$, with $\sigma\circ\sigma$ being the identity mapping, and (b) that multiplication by $e^{i\phi}$ does not destroy membership in $\mathcal{S}$]. Moreover, they have degree $\leq n$, they assume identical values at the points $\mathcal{Z}$, and they correspond to the same trigonometric polynomial $\eta(z)\eta(z)_*$. Yet, they are continuous on the boundary. Thus, by the first part of the proof, they are identical. □

Remark 2: The exposition in this note focuses on positive-real functions (class $\mathcal{C}$) on the unit disc, while an entirely analogous theory holds for the case of bounded-real functions (Schur class $\mathcal{S}$, or contractive), in the disc or on the half-plane.

IV. Conclusion

Analytic interpolation theory is being used extensively in a variety of engineering fields (robust control, $\mathcal{H}_\infty$-control, approximation, circuit theory, signal processing). In mathematics, it has a long history going back to the beginning of the century. However, questions regarding degree constraints were only raised in the engineering literature [12], [9]. The degree of interpolants relates to the dimension of dynamical systems, sought as solutions to engineering problems (e.g., a modeling filter for a stochastic process with given covariance data, or, of a filter satisfying given frequency domain performance objectives). Early results on characterizing interpolants of a fixed degree were obtained in [6]–[8]. The present work was motivated by the recent advances obtained in the research program by Byrnes, Lindquist and their co-workers, e.g., [1]–[3].

REFERENCES

[1] C. I. Byrnes, A. Lindquist, S. V. Gusev, and A. S. Matveev, "A complete parametrization of all positive rational extensions of a covariance sequence," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1841–1857, 1995.

[2] C. I. Byrnes and A. Lindquist, "On the partial stochastic realization problem," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1049–1070, 1997.

[3] C. I. Byrnes, H. J. Landau, and A. Lindquist, "On the well-posedness of the rational covariance extension problem," in *Current and Future Directions in Applied Mathematics*, M. Alber, B. Hu, and J. Rosenthal, Eds. Boston, MA: Birkhäuser, 1997, pp. 83–106.

[4] C. I. Byrnes, S. V. Gusev, and A. Lindquist, "A convex optimization approach to the rational covariance extension problem," preprint.

[5] C. I. Byrnes, T. Georgiou, and A. Lindquist, "A generalized entropy criterion for rational Nevanlinna-Pick interpolation: A convex optimization approach to certain problems in systems and control," to be published.

[6] T. Georgiou, "Partial realization of covariance sequences," Ph.D. thesis, CMST, Univ. of Florida, Gainesville, 1983.

[7] T. Georgiou, "A topological approach to Nevanlinna-Pick interpolation," *SIAM J. Math. Anal.*, vol. 18, pp. 1248–1260, Sept. 1987.

[8] T. Georgiou, "Realization of power spectra from partial covariance sequences," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-35, pp. 438–449, 1987.

[9] R. E. Kalman, "Realization of covariance sequences," in *Toeplitz Centennial*, I. Gohberg, Ed. Boston, MA: Birkhauser Verlag, 1982, pp. 331–342.

[10] D. Sarason, "Generalized interpolation in $\mathcal{H}^\infty$," *Trans. American Math. Society*, vol. 127, pp. 179–203, 1967.

[11] J. L. Walsh, "Interpolation and approximation by rational functions in the complex domain," *American Math. Society*, Colloquium Publications, Providence, RI, 1956, vol. 20, p. 382.

[12] D. C. Youla and M. Saito, "Interpolation with positive-real functions," *J. Franklin Institute*, vol. 284, pp. 77–108, 1967.

A Local Form of Small Gain Theorem and Analysis of Feedback Volterra Systems

Qingsheng Zheng and Evanghelos Zafiriou

***Abstract*—The requirement of evaluating a gain over the entire signal space is one of the restrictions in the traditional small gain theorem. In this paper, a local form of small gain theorem is presented. It yields a bound on the external signal that guarantees that the magnitude of the specified signal along the closed loop stays within a certain region and hence it is useful in addressing the signal magnitude dependent stability problem. The theorem is used to analyze the feedback properties of a Volterra series system as well as an inverse (or pseudo inverse) Volterra system. Improvement over existing results is demonstrated, both theoretically and via numerical examples.**

***Index Terms*— Contractive causality, feedback Volterra system, local small gain theorem, nonlinear system inversion, Volterra series.**

I. INTRODUCTION

The small gain theorem plays a fundamental role in the analysis of nonlinear feedback systems using input–output formulations. It was first proposed in [10] and [17], and comprehensively discussed later (e.g., [2]). It has found wide applications in showing bounded-input bounded-output stability of nonlinear feedback systems, such as in nonlinear internal model control [3] and robust nonlinear control [13]. In its traditional form, it poses some restrictions on its application. One, as pointed out in [5], is that the affine gain formulation can inhibit adoption of input–output stability methods, and so a generalized form of the small gain theorem was proposed there. Another restriction is the system gain, which is defined as an operator norm over the entire input signal space. Such a system gain may not exist; even if it exists in theory, its computation may be too difficult to carry out. Several process control examples were given in [8] to reveal this restriction. In many cases, a system model is available only for a limited range of its input signal. Then, one is required to analyze the feedback properties based on the limited open-loop information.

Volterra series is one of the major modeling tools for nonlinear dynamic systems [12], [9]. There is a resurgence of interest in using Volterra series in recent years, especially for process control [6]. Regarding the feedback properties of a Volterra series system, Halme and Orava studied the invertibility of a polynomial operator in [4], and DeSantis and Porter discussed the well-posedness of a feedback system with a polynomial plant in [1]. In general, not much progress had been made since then. One possible reason might be that the feedback properties of a Volterra system are intertwined with the convergence problem when one tries to get a Volterra series representation for the closed-loop system, and the latter is extremely difficult to analyze in general (cf. [12], [7]). A possible way to avoid this difficulty is to carry out the feedback analysis based on some open-loop properties, instead of first trying to derive a Volterra series expression for the feedback system.

Manuscript received December 27, 1995; revised May 6, 1997. Recommended by Associate Editor, S. Weiland. This work was supported by the National Science Foundation (PYI CTS-9057292 and EEC 9402384).

The authors are with the Department of Chemical Engineering and Institute for Systems Research, University of Maryland, College Park, MD 20742 USA (e-mail: zafiriou@isr.umd.edu).

Publisher Item Identifier S 0018-9286(99)01316-1.