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006

A CONSTRUCTIVE ALGORITHM FOR SENSITIVITY OPTIMIZATION OF PERIODIC SYSTEMS*

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Abstract. In a recent paper (A. Feintuch, P. P. Khargonekar and A. Tannenbaum, *On the sensitivity minimization problem for linear time-varying periodic systems*, this Journal, 24 (1986), pp. 1076–1085), the problem of weighted sensitivity optimization was considered for linear, discrete-time, periodic time-varying systems. Here we present a constructive algorithm for solving this problem.

Key words. sensitivity minimization, periodic systems, Hankel matrix extension problems

AMS(MOS) subject classifications. 93B50, 30D50

1. Introduction. Zames formulated the weighted sensitivity optimization problem in his seminal paper [14]. Since then, this problem has been thoroughly investigated for linear time-invariant systems by many researchers. We refer the interested reader to the recent survey paper of Francis and Doyle [7] for a good exposition and a complete bibliography. Feintuch and Francis [5] considered this and related problems for linear time-varying systems. Our work is motivated by the recent paper [6] of Feintuch, Khargonekar and Tannenbaum where the problem of weighted sensitivity optimization for linear, discrete-time, periodic time-varying systems is considered. In [6] a formula for minimal weighted sensitivity was derived and the existence of an optimal controller was established. Motivated by possible applications to multirate sampled data systems, here we present a constructive algorithm for the computation of optimal controllers. Our algorithm is based on a simple new way of solving the one step extension problem for finite rank block Hankel matrices. The one-step extension problem for general Hankel operators has been investigated in the masterful work of Adamjan, Arov and Krein [1]. The interested reader is also referred to the recent book [12] by Power for certain related extension problems.

In [10], Khargonekar, Poolla and Tannenbaum showed that to any p-output, m-input, *N-periodic*, causal, linear, discrete-time system, one can associate a pN-output, mN-input causal linear *time-invariant* system with transfer function $P(z)$ such that $P(\infty)$ is (block) *lower triangular.* Indeed, lower triangularity is closely related to causality. Feintuch, Khargonekar and Tannenbaum [6] showed that the weighted sensitivity minimization problem of Zames [14] for periodic systems can be reduced to the following problem: Given a $pN\times mN$ transfer matrix $T(z)$ with no poles on the unit circle, find

$$\mu=\inf\{\|T(z)-V(z)\|_\infty: V(z) \text{ is analytic in the complement of the open unit disc including } \infty \text{ and } V(\infty) \text{ is block lower triangular}\}. \tag{1.1}$$

This reduction is accomplished using coprime factorizations, Youla parametrization of all stabilizing controllers, and inner-outer factorizations. (There exist good algorithms for these factorizations, e.g., see Doyle [4], Khargonekar and Sontag [11], Vidyasagar [13], and the references cited there.)

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A formula for μ in (1.1) was given by Feintuch, Khargonekar and Tannenbaum [6]. This formula shows that μ is the maximum of the norms of N Hankel operators. Our paper is devoted to a constructive algorithm to obtain $V(z)$ to solve (1.1) for the special case of rational $T(z)$. This algorithm combined with techniques for coprime and inner-outer factorizations gives a complete constructive algorithm for obtaining optimal controllers for weighted sensitivity minimization of periodic systems.

2. Main results. It has been shown by Feintuch, Khargonekar and Tannenbaum [6] that the weighted sensitivity minimization of Zames [14] for N-periodic linear time-varying finite-dimensional plants can be reduced to the following *best approximation* problem: Given a (possibly unstable) rational $pN\times mN$ matrix $T(z)$, find

$$\mu=\inf\{\|T(z)-V(z)\|_\infty\colon V(z) \text{ with entries in } RH_\infty^{pN\times mN} \text{ such that } V(\infty) \text{ is lower block triangular}\}. \tag{1.1}$$

It is assumed that $T(z)$ has no poles on the unit circle. Here RH_∞ denotes the space of all rational functions with real coefficients which are analytic in the complement of the open unit disc (including infinity). Each $pN\times mN$ matrix is considered as an $N\times N$ square matrix with $p\times m$ block entries. The key constraint here is that $V(\infty)$ is required to be block lower triangular. (This corresponds to causality; see [6].) A formula for μ was given by Feintuch, Khargonekar and Tannenbaum [6] and it was also shown that a $V(z)$ achieving the minimum exists. Our main result is to give a *constructive algorithm* to obtain $V(z)$ which in turn can be used to obtain the optimal controller. We should also note that a solution to this problem will also be a key step in solving the general H^∞-optimization problem of Doyle [4] in the setting of periodic systems.

Let

$$T(z)=\sum_{j=-\infty}^{\infty}\gamma_j z^{-j}$$

be the Fourier series expansion of $T(z)$ (which converges on an open set containing the unit circle). We are seeking a function

$$V(z)=\sum_{j=0}^{\infty}v_j z^{-j}$$

in $RH_\infty^{pN\times mN}$ such that v_0 is lower $(p\times m)$-block triangular and $\|T-V\|_\infty$ is minimized. *Our solution is to first obtain v_0 with the required constraint, and then obtain the rest of $V(z)$ using Glover's algorithm* [9].

From the work of Adamjan, Arov and Krein [1] we know that

$$\inf_{\tilde V}\|T(z)-v_0-\tilde V(z)\|_\infty$$

where $\tilde V(z)=\sum_{j=1}^{\infty}v_j z^{-j}$ is in $z^{-1}RH_\infty^{pN\times mN}$, is equal to the norm of the Hankel operator.

$$\Gamma_e=\begin{bmatrix} D & \gamma_{-1} & \gamma_{-2} & \cdots \\ \gamma_{-1} & \gamma_{-2} & \gamma_{-3} & \cdots \\ \gamma_{-2} & \gamma_{-3} & \gamma_{-4} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \text{where } D=\gamma_0-v_0.$$

The operator Γ_e is thought of as a bounded linear operator acting between the Hilbert spaces of square summable one-sided sequences-denoted by h_2:

$$\Gamma_e: h_2^{mN}\to h_2^{pN}.$$

This result is the matrix version of the Nehari problem. In case $T(z)-v_0$ is a rational matrix valued function, a constructive procedure to obtain a rational $\tilde{V}(z)$ minimizing $\|T(z)-v_0-\tilde{V}(z)\|_\infty$ is given by Glover [9], and Ball and Ran [2]. Thus, we only need to consider a one-step extension problem for the finite-rank Hankel operator given by the matrix $\Gamma=\lfloor\gamma_{1-j-k}\rfloor_1^\infty$. *We want to determine an "extension" D in $R^{pN\times mN}$, subject to the given constraint that the upper block triangular part of D is specified, so that $\|\Gamma_e\|$ is minimized.* In the process of solving the generalized Nehari problem, Adamjan, Arov and Krein [1] have provided a solution to a one-step extension problem for block Hankel operators. However their solution does not seem to lend itself either to an easily computable scheme in the case of finite-rank Hankel operators, or to a procedure dealing with the case where D is paritially specified as is required in our case.

It is a standard result in realization theory (see the book by Fuhrmann [8]) that if Γ is a finite-rank Hankel operator, then there exists a triple of matrices (F, G, H), where F is square $n\times n$ (n being the smallest such integer possible), H is $pN\times n$, and G is $n\times mN$, such that the entries γ_{-k} admit a factorization

$$\gamma_{-k}=HF^{k-1}G, \qquad k=1,2,\cdots. \tag{2.1}$$

Moreover, this induces a factorization of Γ into a product $\mathbf{OR}$, where

$$\mathbf{R}=\lfloor G\quad FG\quad F^2G\cdots\rfloor: h_2^{mN}\to R^n: (u_i: i=0,1,\cdots)\to\sum_{i=0}^{\infty}F^iGu_i,$$

and

$$\mathbf{0}=\begin{bmatrix}H\\HF\\HF^2\\\vdots\end{bmatrix}: R_n\to h_2^{pN}: x\to(HF^ix: i=0,1,\cdots),$$

are bounded linear maps. These are the usual *reachability* and *observability* maps and, because of minimality, are surjective and injective respectively. In view of this factorization, Γ_e is given by

$$\Gamma_e=\begin{bmatrix}D & H\mathbf{R}\\ \mathbf{0}G & \mathbf{0}F\mathbf{R}\end{bmatrix}. \tag{2.2}$$

Let P, Q, Σ, Δ be defined as follows:

$$P:=\mathbf{RR}^*,\quad \mathbf{Q}:=\mathbf{0}^*\mathbf{0},\quad \Sigma:=P^{1/2},\quad \Delta:=Q^{1/2}, \tag{2.3}$$

where $(\ )^{1/2}$ denotes the "Hermitian square root of," and $(\ )^*$ denotes the "adjoint of." Also define the following *finite matrix.*

$$H_e:=\begin{bmatrix}D & H\Sigma\\ \Delta G & \Delta F\Sigma\end{bmatrix}.$$

We now have the following proposition.

PROPOSITION 2.4. *With the above notation* $\|\Gamma_e\|=\|H_e\|$.

Proof. Define

$$\mathbf{U}:=\begin{bmatrix}I_{mN} & 0\\ 0 & \mathbf{R}^*\Sigma^{-1}\end{bmatrix}: R^{mN}+R^n\to R^{mN}+h_2^{mN},$$

$$\mathbf{V}:=\begin{bmatrix}I_{pN} & 0\\ 0 & \Delta^{-1}\mathbf{0}^*\end{bmatrix}: R^{pN}+h_2^{pN}\to R^{pN}+R^n,$$

where I_k denotes the $k\times k$ identity matrix. (That the indicated inverses exist follows easily from the fact that $\mathbf{R}$ is surjective and $\mathbf{0}$ is injective.) It is easily seen that

$$\pi_{\mathbf{R}^*} := \mathbf{R}^*(\mathbf{R}\mathbf{R}^*)^{-1}\mathbf{R}: h_2^{mN} \to h_2^{mN}$$

is the orthogonal projection onto the range of $\mathbf{R}^*$, and that

$$\pi_{\mathbf{0}} := \mathbf{0}(\mathbf{0}^*\mathbf{0})^{-1}\mathbf{0}^*: h_2^{pN} \to h_2^{pN}$$

is the orthogonal projection onto the range of $\mathbf{0}$. It is now straightforward to verify that

$$\mathbf{U}\mathbf{U}^* = I_{mN} + \pi_{\mathbf{R}^*}, \tag{2.5a}$$

$$\mathbf{U}^*\mathbf{U} = I_{mN+n}, \tag{2.5b}$$

$$\mathbf{V}^*\mathbf{V} = I_{pN} + \pi_{\mathbf{0}}, \tag{2.5c}$$

$$\mathbf{V}\mathbf{V}^* = I_{pN+n}. \tag{2.5d}$$

We now prove that $\Gamma_e = \mathbf{V}^* H_e \mathbf{U}^*$. Note first that $H_e = \mathbf{V}\Gamma_e\mathbf{U}$. Then

$$\begin{aligned} \mathbf{V}^* H_e \mathbf{U}^* &= \mathbf{V}^*\mathbf{V}\Gamma_e\mathbf{U}\mathbf{U}^* \\ &= \begin{bmatrix} I_{pN} & 0 \\ 0 & \pi_{\mathbf{0}} \end{bmatrix} \begin{bmatrix} D & H\mathbf{R} \\ \mathbf{0}G & \mathbf{0}F\mathbf{R} \end{bmatrix} \begin{bmatrix} I_{mN} & 0 \\ 0 & \pi_{\mathbf{R}^*} \end{bmatrix} \\ &= \begin{bmatrix} D & H\mathbf{R}\pi_{\mathbf{R}^*} \\ \pi_{\mathbf{0}}\mathbf{0}G & \pi_{\mathbf{0}}\mathbf{0}F\mathbf{R}\pi_{\mathbf{R}^*} \end{bmatrix}. \end{aligned}$$

But $\pi_{\mathbf{0}}\mathbf{0} = \mathbf{0}$, and $\mathbf{R}\pi_{\mathbf{R}^*} = \mathbf{R}[\mathbf{R}^*(\mathbf{R}\mathbf{R}^*)^{-1}\mathbf{R}] = \mathbf{R}$. Consequently,

$$\mathbf{V}^* H_e U^* = \begin{bmatrix} D & H\mathbf{R} \\ \mathbf{0}G & \mathbf{0}F\mathbf{R} \end{bmatrix} = \Gamma_e.$$

Finally, from (2.5b) and (2.5d) it follows that

$$\|\Gamma_e\| = \|H_e\|,$$

and this completes the proof. □

Thus, our original problem has now become: *Obtain D subject to the original constraints, so that it minimizes $\|H_e\|$*. Note that the entries of H_e are now finite matrices, and moreover, the quantities Σ, Δ can be obtained from (F, G, H) by first computing P and Q as solutions to the following Lyapunov equations (see [9]):

$$P = FPF^* + GG^*, \tag{2.6a}$$

$$Q = F^*QF + H^*H, \tag{2.6b}$$

and then taking the Hermitian square roots of P and Q (see (2.3)). Below, we focus on how to explicitly compute such a D, given F, G, H, Σ, Δ.

Therefore, our problem has now been reduced to obtain a $pN\times mN$ matrix D whose $(p\times m)$-block entries above the main diagonal are completely specified by γ_0, and which minimizes

$$\|H_e\| = \left\| \begin{bmatrix} D & H\Sigma \\ \Delta G & \Delta F\Sigma \end{bmatrix} \right\|.$$

Let D be represented by

$$D = \begin{bmatrix} x_{11} & d_{12} & d_{13} & \cdots & d_{1N} \\ x_{21} & x_{22} & d_{23} & \cdots & d_{2N} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & x_{N3} & \cdots & x_{NN} \end{bmatrix},$$

where the entries indicated are $p\times m$ matrices, the d's being specified by γ_0 and the x's representing the entries to be filled in. Define

$$\hat{H}_e := \begin{bmatrix} \Delta G & \Delta F\Sigma \\ D & H\Sigma \end{bmatrix},$$

$$L_k := \begin{bmatrix} I_{n+kp} & 0 \\ 0 & 0 \end{bmatrix},$$

$$R_k := \begin{bmatrix} 0 & 0 \\ 0 & I_{n+km} \end{bmatrix}.$$

We now have the following proposition.

PROPOSITION 2.7. $\mu = \max\{\|L_1\hat{H}_eR_{N-1}\|, \cdots, \|L_N\hat{H}_eR_0\|\}$.

The matrix $L_K\hat{H}_eR_{N-k}$ is the *top right* $(n+km)\times(n+Nm-km)$ submatrix of $\hat{H}_e$. Note that $L_k\hat{H}_eR_{N-k}$, for $k=0,1,\cdots,N$ are all the maximal rectangular submatrices of $\hat{H}_e$ whose entries do not depend on the variables x. (It will be seen that the norm of $L_0\hat{H}_eR_n$ is equal to the norm of $L_N\hat{H}_eR_0$, and this is why only one of the two appears in the expression of the above proposition.) The proof is based on the following very important result—see the book by Power [12] and also [3] for a proof of this result.

LEMMA 2.8 (Parrott, Davis-Kahan-Weinberger). *Consider the block matrix*

$$M(X) = \begin{bmatrix} C & A \\ X & B \end{bmatrix},$$

where A, B, C, X are matrices of compatible dimensions. Then

$$\alpha = \inf_X \|M(X)\| = \max\left\{\|(C\ \ A)\|, \left\|\begin{pmatrix} A \\ B \end{pmatrix}\right\|\right\}.$$

Moreover, the above infimum is attained by the choice

$$X = -BA^*(\alpha^2 I - AA^*)^{-1}C. \tag{2.9}$$

In case the indicated inverse does not exist, then this should be interpreted as a pseudo-inverse. Also, in [3], one can find a description of all possible choices for X that attain this infimum.

Proof of Proposition 2.7. By the results of Adamjan, Arov and Krein [1], as we mentioned earlier, it follows that

$$\mu = \inf_{x_{ik}} \|\Gamma_e\|.$$

From Proposition 2.4 we now have that

$$\mu = \inf_{x_{ik}} \|H_e\| = \inf_{x_{ik}} \|\hat{H}_e\|,$$

where the last equality follows from the unitary equivalence of H_e and $\hat{H}_e$. By repeated application of Lemma 2.8 we now have that

$$\begin{aligned}\mu &= \max\{\|(L_N\hat{H}_eR_0)\|, \inf_{\substack{x_{j1},\cdots,x_{jj}\\ 1\leqq j\leqq N-1}} \|(L_{N-1}\hat{H}_eR_N)\|\}\\ &\ \ \vdots\\ &= \max\{\|(L_N\hat{H}_eR_0)\|, \cdots, \|L_2\hat{H}_eR_{N-2}\|, \inf_{x_{11}} \|L_1\hat{H}_eR_N\|\}\\ &= \max\{\|(L_N\hat{H}_eR_0)\|, \|(L_{N-1}\hat{H}_eR_1)\|, \cdots, \|(L_0\hat{H}_eR_N)\|\}.\end{aligned}$$

A final point to be noted is that the first and the last term of the above expression are equal. We have

$$\|(L_N\hat{H}_eR_0)\| = \left\|\begin{pmatrix}\Delta F\Sigma \\ H\Sigma\end{pmatrix}\right\| = \left\|\begin{pmatrix}H\Sigma \\ \Delta F\Sigma\end{pmatrix}\right\|.$$

But

$$\mathbf{V}^*\begin{bmatrix}H\Sigma \\ \Delta F\Sigma\end{bmatrix}\Sigma^{-1}\mathbf{R} = \begin{bmatrix}H\mathbf{R} \\ 0F\mathbf{R}\end{bmatrix} = \Gamma,$$

and

$$\mathbf{V}\mathbf{V}^* = I_{n+pN} \quad \text{and} \quad \Sigma^{-1}\mathbf{R}\mathbf{R}^*\Sigma = I_n.$$

Consequently, $\|L_N\hat{H}_eR_0\| = \|\Gamma\|$. It follows similarly that $\|L_0\hat{H}_eR_N\| = \|\Gamma\|$, and this completes the proof. □

It can be readily shown that the matrices

$$L_k\hat{H}_eR_{N-k}, \qquad k = 1, \cdots, N,$$

are directly related to the operators $A_k, k = 1, \cdots, N$ respectively, of Feintuch, Khargonekar and Tannenbaum [6], and that in fact they have equal norms. Thus, in Proposition 2.7, we have a formula for the optimal sensitivity μ, that is equivalent to the one given in [6]. But, in addition to that, the sequence of steps (2.9) in the proof of the Proposition 2.7 leads to the following procedure to obtain a $V(z)$ that solves the problem (1.1).

ALGORITHM.

1. Obtain a minimal realization (F, G, H) of the negative Fourier coefficients $(\gamma_k: k \leqq -1)$ of $T(z)$; i.e., (F, G, H) satisfy (2.1).

2. Obtain P, Q by solving Lyapunov equations (2.6a) and (2.6b) and find their Hermitian square roots Σ, Δ, respectively.

3_1. Select x_{11} using (2.9) to minimize $\|L_1\hat{H}_eR_N\|$. Note that in $L_1\hat{H}_eR_N$, the only variable is x_{11} and the rest of $L_1\hat{H}_eR_N$ is completely specified.

3_2. Select (x_{21}, x_{22}) to minimize $\|L_2\hat{H}_eR_N\|$. Note that $L_2\hat{H}_eR_N$ contains x_{11}, x_{21}, x_{22} as the only variables. In this step x_{11} obtained in step 3_1 is used. Again (2.9) is used to select (x_{21}, x_{22}).

3_j. Select $(x_{j1}, x_{j2}, \cdots, x_{jj})$ to minimize $\|L_j\hat{H}_eR_N\|$ for $j = 3, 4, \cdots, N$, where the entries $x_{k1}, \cdots, x_{kk}$, for $1 \leqq k \leqq j-1$ have already been determined at the previous steps. This is done by applying (2.9) to the corresponding submatrices of $L_j\hat{H}_eR_N$.

4. Let $v_0 = \gamma_0 - D$. Now using techniques of Glover [9] obtain

$$\tilde{V}(z) = \sum_{i=1}^{\infty} v_i z^{-1}$$

such that

$$\mu = \|T(z) - v_0 - \tilde{V}(z)\|_\infty.$$

For this last step, see also Ball and Ran [2]. Then $V(z) = v_0 + \tilde{V}(z)$ is a solution to (1.1).

We would like to note that only the elements of D are calculated by recursively applying (2.9) $(N-1)$ times. After obtaining D, the procedure of Glover [9] or Ball and Ran [2] can be used to obtain $V(z)$.

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