On the Computation of Switching Surfaces in Optimal Control: A Gröbner Basis Approach

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***Abstract*—A number of problems in control can be reduced to finding suitable real solutions of algebraic equations. In particular, such a problem arises in the context of switching surfaces in optimal control. Recently, a powerful new methodology for doing symbolic manipulations with polynomial data has been developed and tested, namely the use of Gröbner bases. In this paper, we apply the Gröbner basis technique to find effective solutions to the classical problem of time-optimal control.**

***Index Terms*—Computation algebraic geometry, Gröbner bases, optimal control, switching surfaces.**

I. Introduction

Optimal control is one of the most widely used and studied methodologies in modern systems theory. As is well known, time-optimal problems lead to switching surfaces which typically are defined or may be approximated by polynomial equations [1], [10], [14]. The problem of determining on which side a given trajectory is in relation to the switching surface is of course key in developing the control strategy. Since the complexity of the switching surfaces can grow to be quite large, this may become quickly a formidable task. Here is where new techniques from computational algebraic geometry may become vital in efficiently solving this problem. Thus, while there have been a number of interesting more *ad hoc* approaches to the computation of switching surfaces (see [1], [10], [14], and the references therein), we feel that the techniques presented here can systematize the calculations.

More precisely, in this paper we would like to introduce Gröbner bases in the context of optimal control which will reduce the switching surface problem to a combinatorial one. Gröbner bases have already been employed in a number of applications in robotics and motion planning [5], [16]. Here, we would like to propose them as a potentially powerful tool in optimal control. In addition to the computations of switching surfaces, this paper is intended to be of a tutorial nature. Our main purpose is to introduce a fundamental technique in computational geometry in order to solve an important problem in systems.

The contents of this paper are as follows. In Section II, we give the relevant control background. Section III introduces the basic notions of algebraic geometry, elimination theory, and Gröbner bases. In Section IV, these notions are applied to indicate an explicit solution to the time optimal control problem. In Section V, we make some conclusions and indicate the future course of this work.

II. Switching Surfaces in Optimal Control

We focus on the classical problem of time-optimal control for a system consisting of a chain of integrators. It is standard that for such a system, minimum-time optimal control with a bounded input, leads to "bang-bang" control with at most n switchings—n being the order of the system. The control algorithm usually requires explicit determination of the switching surfaces where the sign of the control input changes. Explicit expressions for switching strategy are in all but the simplest cases prohibitively complicated (e.g., see [10] and [14]).

Consider the linear system with saturated control input

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= x_3(t)\\ \dot{x}_3(t) &= u(t), \qquad \text{where } |u(t)| \le 1\end{aligned}$$

and as objective to drive the system from an initial condition $x(0)$ to a target $x(t_f)$, in minimum time t_f. In this case, the Hamiltonian is

$$\mathcal{H} = 1 + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 u.$$

The costate equations become

$$\begin{aligned}\dot{\lambda}_1(t) &= 0\\ \dot{\lambda}_2(t) &= -\lambda_1(t)\\ \dot{\lambda}_3(t) &= -\lambda_2(t) \qquad (1)\end{aligned}$$

while the optimal $u(t)$ is given by $u(t) = -\text{sign}(\lambda_3(t))$.

A closed-form expression for the optimal $u(t)$ as a function of $x(t)$ can be worked out (e.g., [10], see also [14]). Such an expression in fact tests the location of the state vector with regard to a switching surface. Bang-bang switching in practice is not desirable because of the incapacitating effect of noise and chattering. This issue has been addressed by a number of authors (see [14] and the references therein) and will not be discussed

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herein. While various remedies have been proposed and applied, the basic issue of knowing the switching surfaces is still instrumental in most methodologies.

The approach we take herein is algebraic in nature. The idea is to test directly whether a particular switching strategy is feasible. There are only two possible strategies where the input alternates between $+1$ and -1, taking the values $+1, -1, +1, \ldots$, or $-1, +1, -1, \ldots$, respectively. In each case, taking into account the maximal number of switchings, one can easily derive an expression for the final value of the state as a function of the switching times. This expression is then analyzed against the requirement of a given $x(t_f)$.

For this standard time-optimal control problem, it is well known and easy to see by analyzing (1) that, in general, there are no singular intervals, and that the control input switches at most three times. Designate by t_1, t_2 and t_3, the length of the successive intervals where $u(t)$ stays constant. Any set of initial and final conditions can be translated to having $x(0) = 0$ and a given value for $x(t_f)$, and this is the setting from here on. The particular choice (among the only two possible ones)

$$u(t) = \begin{cases} +1, & \text{for } 0 \le t < t_1 \\ -1, & \text{for } t_1 \le t < t_1 + t_2 \\ +1, & \text{for } t_1 + t_2 \le t < t_1 + t_2 + t_3 =: t_f \end{cases}$$

drives the chain of integrators for the origin to the final point $x(t_f)$ given by

$$\begin{aligned} x_3(t_f) &= t_1 - t_2 + t_3 \\ x_2(t_f) &= \frac{t_1^2}{2} + t_1 t_2 - \frac{t_2^2}{2} - t_2 t_3 + \frac{t_3^2}{2} + t_3 t_1 \\ x_1(t_f) &= \frac{t_1^3}{6} - \frac{t_2^3}{6} + \frac{t_3^3}{6} + \frac{t_1^2}{2} t_2 + \frac{t_1^2}{2} t_3 + \frac{t_2^2}{2} t_1 - \frac{t_2^2}{2} t_3 \\ &\quad + \frac{t_3^2}{2} t_1 - \frac{t_3^2}{2} t_2 + t_1 t_2 t_3. \end{aligned} \tag{2}$$

It turns out that the selection between alternating values $+1, -1, +1, \ldots$ or, $-1, +1, -1, \ldots$ for the optimal input $u(t)$ depends on whether the equations in (2) have a solution for a specified final condition $x(t_f) = (x_1, x_2, x_3)'$.

III. COMPUTATIONAL ALGEBRAIC GEOMETRY AND GRÖBNER BASES

Algebraic geometry is concerned with the properties of geometric objects defined as the common zeros of systems of polynomials that are called *varieties*. As such, it is intimately related to the study of rings of polynomials and the associated ideal theory [8], [6].

More precisely, let k denote a field (e.g., the fields of complex numbers $\mathbb{C}$, real numbers $\mathbb{R}$, or rational numbers $\mathbb{Q}$). Over an algebraically closed field such as $\mathbb{C}$, one may show that affine geometry (the study of subvarieties of affine space k^n) is equivalent to the ideal theory of the polynomial ring $k[x_1, \ldots, x_n]$ (see [8], especially the discussion of the Hilbert Nullenstellensatz; see in particular [5, p. 34, Th. 1, 2 pp. 168–170]).

Clearly, the ability to manipulate polynomials and to understand the geometry of the underlying varieties can be very important in a number of applied fields (e.g., the kinematic map in robotics is typically polynomial; see also [15], [16], and the references therein for a variety of applications of geometry to systems theory). We show now how the problem in optimal control discussed above may be reduced to a problem in affine geometry.

Until recently applications of algebraic geometry to practical fields of mathematics was limited because despite its vast number of deep results, very little could actually be effectively computed. Because of this, it has not lived up to its potential to have a major impact on more applied fields. The advent of *Gröbner bases* with powerful fast computers has largely remedied this situation. Gröbner bases were used first by F. Macaulay in his theory of modular systems; he computed with them what is known today as Hilbert functions of Artinian modules. In the 1960s, B. Buchberger defined and named them in honor of his doctoral advisor W. Gröbner. Buchberger also established basic existence theorems and provided an algorithm for computing them, later named after him. They were also essentially discovered by H. Hironaka at around the same time in connection with his work on resolution of singularities. We follow the treatments in [3], [5], and [2].

The method of Gröbner bases helps one to treat a number of key problems for reasonably sized systems of polynomial equations. Among these are the following (see [5, p. 47]).

1) Find all common solutions in k^n of a system of polynomial equations

$$f_1(x_1, \ldots, x_n) = \cdots = f_m(x_1, \ldots, x_n) = 0.$$

2) Determine the (finite set of) generators of a given polynomial ideal.
3) For a given polynomial f and an ideal I, determine whether $f \in I$.
4) Let $g_i(t_1, \ldots, t_m)$, $i = 1, \ldots, n$ be a finite set of rational functions. Suppose $V \subset k^n$ is defined parametrically as $x_i = g_i(t_1, \ldots, t_m)$, $i = 1, \ldots, n$. Find the system of polynomial equations which define the variety V.

A. Gröbner Bases

Gröbner bases generalize the usual Gauss reduction from linear algebra, the Euclidean algorithm in $\mathbb{C}[x]$, and the simplex algorithm from linear programming.

Motivated by the long division in the polynomial ring of one variable, one introduces an order on the monomials in polynomial rings of several variables $k[x_1, \ldots, x_n]$ in order to execute a division type algorithm.

Let ${\mathbb{Z}_+}^n$ denote the set of n-tuples of nonnegative integers. Let $\alpha, \beta \in {\mathbb{Z}_+}^n$. For $\alpha = (\alpha_1, \ldots, \alpha_n)$, and set $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Let $>$ denote a total (linear) ordering on ${\mathbb{Z}_+}^n$ (this means that exactly one of the following statements is true: $\alpha > \beta$, $\alpha < \beta$, or $\alpha = \beta$). Moreover we say that $x^\alpha > x^\beta$ if $\alpha > \beta$. Then, a *monomial ordering* on ${\mathbb{Z}_+}^n$ is a total ordering such that

1) if $\alpha > \beta$ and $\gamma \in {\mathbb{Z}_+}^n$, then $\alpha + \gamma > \beta + \gamma$;
2) $>$ is a *well-ordering*, i.e., every nonempty subset of ${\mathbb{Z}_+}^n$ has a smallest element.

One of the most commonly used monomial orderings is the one defined by the ordinary lexicographical order $>_{lex}$ on ${\mathbb{Z}_+}^n$. Recall that this means $\alpha >_{lex} \beta$ if the leftmost nonzero element of $\alpha - \beta$ is positive. This ordering is also called *elimination order with* $x_1 > \ldots > x_n$.

We now fix a monomial order on ${\mathbb{Z}_+}^n$. Then the *multi-degree* of an element $f = \sum_\alpha a_\alpha x^\alpha \in k[x_1, \ldots, x_n]$ [denoted by multideg(f)] is defined to be the maximum α such that $a_\alpha \neq 0$. The *leading term* of f [denoted by LT(f)] is the monomial

$$a_{\mathrm{multi\,deg}(f)} \cdot x^{\mathrm{multi\,deg}(f)}.$$

We now state the following central definition.

Definition 1: A finite set of polynomials $f_1, \ldots, f_m$ of an ideal $I \subset k[x_1, \ldots, x_n]$ is called a *Gröbner basis* if the ideal generated by LT(f_i) for $i = 1, \ldots, m$ is equal to the ideal generated by the leading terms of all the elements of I, that is

$$\begin{aligned} &k[x_1, \ldots, x_n] \cdot (\mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_m)) \\ &\qquad = k[x_1, \ldots, x_n] \cdot \{\mathrm{LT}(f) \mid f \in I\}. \end{aligned}$$

We emphasize the finiteness of a Gröbner basis.

We remind the reader that the ideal generated by a set of elements in a ring, is the set of all their linear combinations with coefficients taken from the ring. The crucial result on the existence and key property of Gröbner bases is the following.

Theorem 1: Every nontrivial ideal has a Gröbner basis. Moreover, any Gröbner basis of I is a generating set of I.

The *Buchberger algorithm* is a finite algorithm that takes in a finite set of generators for the ideal I in $k[x_1, \ldots, x_n]$ and returns a Gröbner basis for I. At its heart lies the idea of canceling leading terms to obtain polynomials with smaller leading terms, similar to the Euclidean algorithm in $k[x]$. A nice exposition can be found in [5, p. 86].

Notice that the use of Gröbner bases reduces the study of generators of polynomial ideals (and so affine algebraic geometry) to that of the combinatorial properties of monomial ideals. Therein lies the power of this method assuming that one can easily compute a Gröbner basis (see [3] and [5]).

In what follows, we will indicate how Gröbner basis techniques may be used to solve polynomial equations.

B. Elimination Theory

Elimination theory is a classical method in algebraic geometry for eliminating variables from systems of polynomial equations and as such is a key method in finding their solutions. Gröbner bases give a powerful method for carrying out this procedure systematically. We work over an algebraically closed field k in this section.

More precisely, let $I \subset k[x_1, \ldots, x_n]$ be an ideal. The jth *elimination ideal* of I is defined to be

$$I_j = I \cap k\left[x_{j+1}, \ldots, x_n\right].$$

Suppose that I is generated by $f_1, \ldots, f_m$. Then, I_j is the set of all consequences of $f_1 = \cdots = f_m = 0$ which do not involve the variables $x_1, \ldots, x_j$. Thus, elimination of $x_1, \ldots, x_j$ amounts to finding generators of I_j. This is where the Gröbner basis methodology plays the key role.

Theorem 2 (Elimination Theorem): Let $I \subset k[x_1, \ldots, x_n]$ be an ideal, and G a Gröbner basis for I with respect to the lexicographical order with $x_1 > \cdots > x_n$. For every $j = 0, \ldots, n$, set

$$G_j := G \cap k\left[x_{j+1}, \ldots, x_n\right]$$

(i.e., select the elements of G not involving $x_1, \ldots, x_j$). Then, G_j is a Gröbner basis of I_j. (Here, we take $I_0 = I$.)

A proof can be found in [5, p. 113]. Note that, for $l \in \mathbb{Z}^+$, G_j is also a Gröbner basis for $I_{j-l} \cap k[x_{j+1}, \ldots, x_n] = I_j$. Thus, using Theorem 2, we may eliminate the variables one at a time (or all but x_n at once) until we are left with a polynomial in x_n, which we may solve. We must of course then extend the solution to the original system. For an ideal $I \subseteq k[x_1, \ldots, x_n]$ we set

$$V(I) := \{(z_1, \ldots, z_n) \in k^n \colon f(z_1, \ldots, z_n) = 0 \quad \forall f \in I\}.$$

Again, this can be done in a systematic manner via the following result.

Theorem 3 (Extension Theorem): Let $I \subset k[x_1, \ldots, x_n]$ be generated by $f_1, \ldots, f_m$. Let I_1 be the first elimination ideal of I as defined above. For each $i = 1, \ldots, m$, write f_i as

$$f_i = g_i(x_2, \ldots, x_n) x_1^{n_i} + \text{lower order terms in } x_1$$

where n_i is the largest exponent of x_1. Suppose that $(z_2, \ldots, z_n) \in V(I_1) \subseteq k^{n-1}$. If there exists some i such that $g_i(z_2, \ldots, z_n) \neq 0$, then we may extend $(z_2, \ldots, z_n)$ to a solution of $(z_1, \ldots, z_n) \in V(I)$.

The theorem gives a systematic way of checking whether partial solutions of I_j may be extended to solutions of I. A detail discussion and proof can be found in [5, p. 115].

This ends our brief discussion of Gröbner bases and elimination theory. We should note that there are symbolic implementations of this methodology on such standard packages as Mathematica, Maple, or Macaulay [11].

IV. COMPUTATION OF SWITCHING SURFACES

In this section, we indicate the solution to the time optimal control problem formulated in Section II. Even though we work out the case of third-order system, the method we propose is completely general, and should extend in a straightforward manner to any number of switchings.

In what follows below, we set

$$\begin{aligned} x &:= t_1 \\ y &:= t_2 \\ z &:= t_3 \end{aligned}$$

and

$$\begin{aligned} a &:= x_3(t_f) \\ b &:= x_2(t_f) \\ c &:= x_3(t_f). \end{aligned}$$

A. Complex Solutions

In this subsection, we solve the complex version of the switching problem, namely, the following.

Problem 1: Given is the system of equations

$$
\begin{gathered}
x - y + z = a \\
\frac{x^2}{2} + xy + \frac{z^2}{2} + zx - \frac{y^2}{2} - yz = b \\
\frac{x^3}{6} + \frac{z^3}{6} + \frac{x^2y}{2} + \frac{x^2z}{2} + \frac{y^2x}{2} + \frac{z^2x}{2} \\
+ xyz - \frac{y^3}{6} - \frac{y^2z}{2} - \frac{z^2y}{2} = c.
\end{gathered} \tag{3}
$$

We are interested in solving the following question: if $a, b, c \in \mathbb{C}$, does the system have complex solutions x, y, z? The answer will be yes.

To illustrate the use of the Macaulay symbolic program in computational algebraic geometry, we will put in some of the relevant scripts. Let us call I the ideal in $\mathbb{Q}[x, y, z, a, b, c]$ generated by the three forms above. As a first step, let us compute a Gröbner basis for I. We introduce the elimination order with $x > y > z > c > b > a$. Here is a Macaulay command sequence to accomplish this.

```
1% ring R
! characteristic (if not 31 991)     ?
! number of variables                ? 6
!   6 variables, please              ? xyzcba
! variable weights (if not all 1)    ?
! monomial order (if not rev. lex.)  ? 1 1 1 1 1 1
  largest degree of a monomial
    : 512 512 512 512 512 512
1% < ideal I x − y + z − a
     x2/2 + xy + z2/2 + zx − y2/2 − yz − b\
   x3/6 + z3/6 + x2y/2 + x2z/2 + y2x/2 + z2x/2 + xyz
     −y3/6 − y2z/2 − z2y/2 − c
1% < inhomog_std I II
```

The result is the following seven forms, made visible by

```
putstd II
```

$$
\begin{aligned}
& z^4b - 1/2z^4a^2 - 2z^3c - 2z^3ba + 4/3z^3a^3 + 6z^2ca \\
&\quad + z^2b^2 - z^2ba^2 - 3/4z^2a^4 - 4zcb - 4zca^2 + 2zb^2a \\
&\quad + 2/3zba^3 + 1/6za^5 + c^2 + 2cba + 2/3ca^3 - b^3 \\
&\quad - 1/2b^2a^2 - 1/12ba^4 - 1/72a^6,
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
& yc^2 - 2ycba + 2/3yca^3 + yb^3 - 1/2yb^2a^2 + 1/12yba^4 \\
&\quad - 1/72ya^6 + z^3b^2 - z^3ba^2 + 1/4z^3a^4 - z^2cb \\
&\quad + 1/2z^2ca^2 - 2z^2b^2a + 13/6z^2ba^3 - 7/12z^2a^5 \\
&\quad - 2zc^2 + 6zcba - 7/3zca^3 - zb^2a^2 + 7/36za^6 \\
&\quad + 2c^2a - 2cb^2 - 3cba^2 + 4/3ca^4 + 2b^3a \\
&\quad - 2/3b^2a^3 - 1/36a^7
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
& yzb - 1/2yza^2 - yc + 1/6ya^3 - z^2b + 1/2z^2a^2 + 2zc \\
&\quad - 1/3za^3 - 2ca + b^2 + 1/12a^4
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
& yzc - 1/6yza^3 - 2yca + yb^2 + 1/12ya^4 + z^3b - 1/2z^3a^2 \\
&\quad - 2z^2c - 2z^2ba + 4/3z^2a^3 + 6zca - zba^2 - 1/2za^4 \\
&\quad - cb - 7/2ca^2 + 2b^2a + 1/6ba^3 + 1/12a^5
\end{aligned} \tag{7}
$$

$$
yz^2 - 2yza + yb + 1/2ya^2 + zb - 1/2za^2 - c + 1/6a^3 \tag{8}
$$

$$
y^2 - 2yz + 2ya - b + 1/2a^2 \tag{9}
$$

$$
x - y + z - a. \tag{10}
$$

We read these equations as polynomials in x, y, z with parametric coefficients that depend on a, b, c.

At this point we remark that the Gröbner basis would look just the same if we had considered the extension of the ideal to the ring of polynomials over $\mathbb{R}$ or $\mathbb{C}$. This is true in general.

Now, if the three forms from Problem 1 have a solution, then certainly the quartic given by (4) above, also must have a solution, whatever the base field. Over $\mathbb{C}$ this will have a solution for sure if the leading form is nonzero, which is the case if and only if $a^2 - 2b \neq 0$.

Moreover, if the quartic (4) does indeed have a solution over $\mathbb{C}$ (i.e., $\exists z \in \mathbb{C}$ that makes the equation true for chosen $a, b, c \in \mathbb{C}$), then the Extension Theorem tells us, in view of (9) and(10), that we can find y and then x in $\mathbb{C}$ solving the entire system over $\mathbb{C}$.

Let us continue to investigate the question whether we can find $z \in \mathbb{C}$ such that the quartic holds true in the case where $a^2 = 2b$. In that case, we need to add $a^2 - 2b$ to the generators of our ideal, and recompute the Gröbner basis. Here is the script

```
1% < ideal J a^2 − 2b
1% concat J I
1% < inhomog_std J JJ
```

In this case the output is

$$
\begin{aligned}
& b - 1/2a^2 \\
& z^3c - 1/6z^3a^3 - 3z^2ca + 1/2z^2a^4 + 3zca^2 \\
&\quad - 1/2za^5 - 1/2c^2 - 5/6ca^3 + 11/72a^6 \\
& yc - 1/6ya^3 - 2zc + 1/3za^3 + 2ca - 1/3a^4 \\
& yz^2 - 2yza + ya^2 - c + 1/6a^3 \\
& y^2 - 2yz + 2ya \\
& x - y + z - a.
\end{aligned}
$$

Not surprisingly, the quartic became a cubic when we set the leading coefficient to zero. As before, the cubic will have a complex root as long as the leading coefficient $a^3 - 6c$ is nonzero. Also, as before, the two last equations ensure that each solution for z may be extended to (x, y, z) solving the system.

What happens if $a^3 = 6c$? Let us add this relation and recompute a Gröbner basis

```
1% < ideal K a^3 − 6c
1% concat K J
1% < inhomog_std K KK
```

which leads to

$$\begin{gathered} b - 1/2a^2 \\ c - 1/6a^3 \\ yz^2 - 2yza + ya^2 \\ y^2 - 2yz + 2ya \\ x - y + z - a. \end{gathered}$$

A somewhat surprising thing happened: when we killed the leading coefficient of the cubic, the entire polynomial died. Let us factor as much as we can in the output

$$\begin{gathered} b - 1/2a^2 \\ c - 1/6a^3 \\ y(z-a)^2, \\ y(y - 2(z-a)) \\ x - y + (z - a). \end{gathered}$$

One can see that this system has, for example, the solution $(x, y, z) = (0, 0, a)$.

We conclude that

1) the system does always have a complex solution;
2) if $a^2 = 2b$, $a^3 = 6c$ and $a, b, c \in \mathbb{R}$, the system has real solutions;
3) if $a^2 = 2b$, $a^3 = 6c$ and $0 \le a, b, c \in \mathbb{R}$, the system has real nonnegative solutions.

B. Real Positive Solutions

Now that we have established the existence of complex solutions x, y, z for any parameter set (a, b, c) let us search for the existence of real nonnegative solutions for real parameters. This will solve our switching control problem. Thus, as a second step we will answer the following.

Problem 2: Given are $a, b, c \in \mathbb{R}$. Does there exist a nonnegative solution vector (x, y, z) for the system (3) in the sense that $x \ge 0, y \ge 0, z \ge 0$?

Thus, if there is a positive solution x, y, z, then the value of the optimal control u assumes the values $+1, -1, +1$ successively, and in particular, the present value for the optimal control is $u(0) = +1$. If no positive solution exists then the present value of the optimal control is $u(0) = -1$.

The techniques we will use are computations of suitable Gröbner bases together with an algorithm from real algebraic geometry called *Sturm sequences*. Sturm sequences are associated to polynomials as follows. Suppose $f(x)$ is a single variable polynomial with real coefficients. We define $p_0(x) = f(x)$, $p_1(x) = f'(x)$, and then recursively p_i by $p_i = q_{i-1}p_{i-1} - p_{i-2}$ for $i > 1$, where q_{i-1} represents the quotient and p_{i-2} the respective remainder each time. Here, we demand that $\deg(p_i) < \deg(p_{i-1})$. So, p_i is up to sign the remainder of Euclidean division of p_{i-2} by p_{i-1}.

Theorem 2 [4, Th. 1.2]: Let $\alpha < \beta$ be real numbers which are not roots of $f(x)$. Define a function $v(\gamma)$ for $\gamma \in \mathbb{R}$ by counting the number of sign changes in the sequence $\{p_i(\gamma)\}_{i \ge 0}$, dropping all zeros. Then $v(\alpha) - v(\beta)$ is the number of distinct zeros of f between α and β.

The significance of the theorem for us lies in the fact that although it does not specify the location of the zeros it gives a qualitative answer, which as pointed out above is all we need to know about for the purpose of dynamical steering.

As a first step, we compute a Gröbner basis for the three polynomials in (3) under an elimination order with $x > z > y > c > b > a$. Note the switch of the variables y and z in the ordering. One gets

$$\begin{aligned} &y^4 + 4y^2b - 2y^2a^2 - 4yc + 4yba - 4/3ya^3 - b^2 \\ &\quad + ba^2 - 1/4a^4 \qquad (11) \end{aligned}$$

$$zb - 1/2za^2 + 1/2y^3 + 3/2yb - 3/4ya^2 - 2c + ba - 1/6a^3 \qquad (12)$$

$$zy - 1/2y^2 - ya + 1/2b - 1/4a^2 \qquad (13)$$

$$x + z - y - a. \qquad (14)$$

This suggests that one ought to solve (12) or (13) for z

$$z = \frac{-1/2y^3 - 3/2yb + 3/4ya^2 + 2c - ba + 1/6a^3}{b - 1/2a^2} \qquad (15)$$

$$z = \frac{y^2/2 + ya - 1/2b + 1/4a^2}{y} \qquad (16)$$

respectively. This, of course, is assuming that y and $b - a^2/2$ are not zero.

It is easy to check that these solutions for z are not contradicting each other. In fact, they differ by a multiple of the quartic in y, given in (11).

One sees that $y = 0$ implies $2b - a^2 = 6c - a^3 = 0$. These relations simplify the system to

$$\begin{aligned} &b - 1/2a^2 \\ &c - 1/6a^3 \\ &y^3 \\ &zy - 1/2y^2 - ya \\ &x + z - y - a. \qquad (17) \end{aligned}$$

This has the solutions $y = 0$, $z =$ arbitrary, $x = a - z$. Since $y = 0$ is actually equivalent to $a^3 - 6c = a^2 - 2b = 0$, testing the latter conditions is sufficient to find out whether $y = 0$. In that case, nonnegative solutions will exist precisely when a is nonnegative. This covers the case $y = 0$.

If $x = 0$, our system takes the form

$$\begin{aligned} &c^2 - 2cba + 2/3ca^3 + b^3 - 1/2b^2a^2 + 1/12ba^4 - 1/72a^6 \\ &yb - 1/2ya^2 - c + ba - 1/3a^3 \\ &yc - 1/6ya^3 - ca + b^2 - 1/12a^4 \\ &y^2 + b - 1/2a^2 \\ &z - y - a \\ &x. \qquad (18) \end{aligned}$$

Since a, b, c are known, it is easy to check the consistency of this system, by solving each of the three middle equations for y and testing the vanishing of the first. If consistency fails, we are not in the case $x = 0$.

If the system is consistent, one needs to check whether the obtained solutions for y, z are nonnegative. If that is so, set $u = 1$ and otherwise $u = -1$, finishing the case $x = 0$.

In a similar fashion, one does get the case $z = 0$. If $z = 0$, one gets

$$\begin{aligned}
&c^2 + 2cba + 2/3ca^3 - b^3 - 1/2b^2a^2 - 1/12ba^4 - 1/72a^6\\
&yb + 1/2ya^2 - c + 1/6a^3\\
&yc - 1/6ya^3 + 2ca - b^2 - 1/12a^4\\
&y^2 + 2ya - b + 1/2a^2\\
&z\\
&x - y - a \qquad (19)
\end{aligned}$$

which is quite similar to the case $x = 0$. One first checks whether the first relation between the parameters holds. Then one solves the next three equations for y and then solves the last relation for x. If the system is consistent we have $z = 0$. If x, y turn out to be nonnegative set $u = 1$ and otherwise $u = -1$.

This rules out all cases of vanishing variables. In order to predict when strictly positive solutions exist, we are reduced to the cases $(a^2/2 = b, a^3/6 \neq c)$ and $(a^2/2 \neq b)$.

Let us consider first the case $(a^2/2 = b, a^3/6 \neq c)$. Then, we have a Gröbner basis

$$b - 1/2a^2 y^3 - 4c + 2/3a^3 z - y/2 + ax + z - y - a.$$

It becomes obvious that in order to have a nonnegative solution, we need

$$\begin{aligned}
y^3 &= 4\left(c - a^3/6\right) \geq 0\\
z &= \left(4\left(c - a^3/6\right)\right)^{1/3}\Big/2 + a \geq 0\\
x &= \left(4\left(c - a^3/6\right)\right)^{1/3}\Big/2 \geq 0
\end{aligned}$$

which simplifies to the two conditions $c - a^3/6 \geq 0$, $(4(c - a^3/6))^{1/3}/2 + a \geq 0$. These are conditions that can easily be checked for given a, b, c and determine existence of a nonnegative solution (x, y, z) of the system (3).

Now, let us move to the most general situation $b - a^2/2 \neq 0$. In particular, $y \neq 0$ then. Theorem 2 asserts that the Sturm sequence $\{p_i(y)\}$ corresponding to

$$f(y) = y^4 + 4y^2\left(b - a^2/2\right) + 4y\left(ba - c - 1/3a^3\right) - b^2 + ba^2 - a^4/4$$

counts the zeros of this quartic. In particular, there will be positive solutions for just y if and only if $v(0) - v(\infty) > 0$ since zero is not a root of the quartic [note that $-b^2 + ba^2 - a^4/4 = -(b - a^2/2)^2$].

Now, from (9)

$$z = \frac{y^2/2 + ya - 1/2b + 1/4a^2}{y}.$$

This means that for positive y, z is positive as long as $y^2/2 + ya - 1/2b + 1/4a^2 > 0$. This parabola has roots in $r_{1,2} = a \pm \sqrt{b + a^2/2}$ where $r_1 \leq r_2$. Since the parabola has positive leading coefficient, $y, z > 0$ for $y \notin [r_1, r_2]$ if $b + a^2/2 > 0$, and $y, z > 0$ for all $y > 0$ if $b + a^2/2 < 0$.

Similarly

$$x = y + a - z = \frac{y^2/2 + 1/2b - 1/4a^2}{y}.$$

Let $r'_{1,2} = \pm\sqrt{1/2a^2 - b}$ with $r'_1 \leq r'_2$. Hence, $x, y > 0$ if and only if $0 < y \notin [r'_1, r'_2]$ if $a^2/2 > b$, and $x, y > 0$ for all $y > 0$ if $a^2/2 < b$.

We conclude that in order to have x, y, z all positive at the same time we need to satisfy the following conditions all at the same time:

$$\begin{aligned}
&y^4 + 4y^2\left(b - a^2/2\right) + y\left(-4c + 4ba - 4/3a^3\right)\\
&\qquad - b^2 + ba^2 - a^4/4 = 0
\end{aligned}$$

$$\begin{aligned}
&y \notin [r_1, r_2], \quad \text{or} \quad r_i \notin \mathbb{R}\\
&y \notin \left[r'_1, r'_2\right], \quad \text{or} \quad r'_i \notin \mathbb{R}\\
&y > 0
\end{aligned}$$

which can be checked with Sturm sequences.

C. The Switching Algorithm

These results pave the way for the following algorithm. The algorithm has as input the current state (a, b, c) of the system and as output the recommended value for u for time optimal control, either 1 or -1. The origin is then approached by iterated repetition of the algorithm.

Algorithm 3 (Dynamical Steering of the System to the Origin): Suppose our system is in the state (a, b, c).

Case 1) (Check whether $x = 0$.) Test the consistency of the system (18). If consistent solve it; if $y, z \geq 0$ set $u = 1$, otherwise set $u = -1$. If the system (18) is not consistent, go to the next case.

Case 2) (Check whether $z = 0$.) Test the consistency of the system (19). If consistent solve it; if $x, y \geq 0$ set $u = 1$, otherwise set $u = -1$. If the system (19) is not consistent, go to the next case.

Case 3) $2b = a^2, 6c = a^3$. (Check whether $y = 0$.) If $a \geq 0$, set $u = 1$ for a s, at which point the system will have reached the origin. If $a < 0$, let $u = -1$ for a s.

Case 4) $2b = a^2, 6c \neq a^3$; x, y, z *all* $\neq 0$. If $6c - a^3 > 0$ and $6c > -11a^3$, let $u = 1$. Else, let $u = -1$.

Case 5) $2b \neq a^2$, x, y, z *all* $\neq 0$. Set $r_1 = a - \sqrt{b + a^2/2}$, $r_2 = a + \sqrt{b + a^2/2}$, $r'_2 = \sqrt{a^2/2 - b}$. Let $f(y) = y^4 + 4y^2(b - a^2/2) + y(-4c + 4ba - 4/3a^3) - b^2 + ba^2 - a^4/4$ and compute the corresponding Sturm sequence $\{p_i(y)\}_{i \leq 0}$. Let $I = (0, r_1) \cup (r_2, \infty)$ if $r_i \in \mathbb{R}$ and $(0, \infty)$ else. Let $I' = (r'_2, \infty)$ if $r'_2 \in \mathbb{R}$ and $(0, \infty)$ else. Let $S = I \cap I'$.

Using the Sturm sequence compute the number of solutions of $f(y)$ in S. If this number is positive, set $u = 1$ and, otherwise, set $u = -1$.

V. CONCLUSION

This paper has provided a general approach to the switching control strategy in time-optimal control. The key idea is to use the Gröbner basis technique which allows one to algorithmically work with systems of polynomials in several variables. These results are quite general, and we expect that this approach will lead to a complete solution of the problem of identifying switching surfaces, in the sense that we will be able to provide a symbolic computer program which will allow one to solve the problem for a reasonable number of variables (with "reasonable" a function of the computing power of the machine doing the computation).

Gröbner bases are the key tools of computational algebraic geometry. As such, they are expected to have an ever increasing role in the area of systems and control, and in particular, in problems where the solutions form a semialgebraic set whose description hinges upon solving simultaneous polynomial equations and inequalities. Specifically, such areas, in our view (besides time-optimal control and motion planning in robotics, as mentioned earlier) include the characterization of minimal order models from covariance data [12], [13], the classical output feedback problem, decentralized control, and nonlinear control.

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