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State covariances and the matrix completion problem

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***Abstract*— State statistics of a linear system obey certain structural constraints that arise from the underlying dynamics and the directionality of input disturbances. Herein, we formulate completion problems of partially known state statistics with the added freedom of identifying disturbance dynamics. The goal of the proposed completion problem is to obtain information about input excitations that explain observed sample statistics. Our formulation aims at low-complexity models for admissible disturbances. The complexity represents the dimensionality of the subspace of the state-dynamics that is directly affected by disturbances. An example is provided to illustrate that colored-in-time stochastic processes can be effectively used to explain available data.**

***Index Terms*— Convex optimization, low-rank approximation, nuclear norm regularization, state covariances, structured matrix completion problems, noise statistics.**

I. Introduction

The motivation for this work stems from the need to explain statistics of systems with very large number of degrees of freedom with models of low complexity. One such example arises in fluid flows where the dynamics are governed by the Navier-Stokes (NS) equations. Although, in principle, these equations capture all features of the underlying dynamics they are prohibitively complex for model-based analysis and design. It has been proposed and successfully demonstrated that linearized NS equations in the presence of stochastic excitation are sufficient to qualitatively predict structural features of wall-bounded shear flows [1]–[4]. In all prior studies excitations have been restricted to white-in-time stochastic processes. This assumption is often too restrictive to explain observed statistics of turbulent flows. Therefore, our interest is in developing a framework to allow for the more general class of colored-in-time stochastic disturbances.

The data for our problem comes in the form of partially available second-order statistics. These are typically generated in experiments or high-fidelity numerical simulations and, thus, our aim is to reproduce statistics with linear dynamics with known generator. The generator arises from linearization around given equilibrium profile. On the other hand, neither the way disturbance enters into the state equation, nor its power spectrum are known. Since white-in-time disturbances may be insufficient to account for observed statistics, we examine the possibility to explain data using disturbances with non-trivial power spectra.

The structure of state covariances for colored-in-time disturbances has been studied in [5], [6]. This framework naturally leads us to formulate a new class of matrix completion problems. Complexity of the disturbance model can be expressed in terms of rank. We study structural relations and constraints between the parameters of the disturbance model and the requirement that sampled covariances are reproducible by linear dynamics. We utilize the nuclear norm as a proxy for the rank [7]–[11], and formulate convex optimization problems to address our modeling paradigm.

The paper is organized as follows. In Section II we briefly summarize key results on the structure of state covariances. In Section III we characterize admissible signatures for certain Hermitian matrices that parametrize disturbance spectra and provide the theoretical basis for the rank minimization problem that we formulate in Section IV. To highlight theory and concepts we give an example in Section V and recap with concluding remarks in Section VI.

II. State covariances

Consider a linear time-invariant system

$$\dot{x} \;=\; Ax \;+\; Bu \tag{1}$$

where $x(t) \in \mathbb{C}^n$ is a state vector, $u(t) \in \mathbb{C}^m$ is a zero-mean stationary stochastic input, $A \in \mathbb{C}^{n\times n}$ is a Hurwitz matrix, $B \in \mathbb{C}^{n\times m}$ with $m \leq n$ is a full column rank matrix, and (A, B) is a controllable pair. The steady-state covariance Σ of the state vector in (1) satisfies

$$\operatorname{rank}\begin{bmatrix} A\Sigma + \Sigma A^* & B \\ B^* & 0 \end{bmatrix} \;=\; \operatorname{rank}\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}. \tag{2a}$$

Equivalently, the equation

$$A\Sigma \,+\, \Sigma A^* \;=\; -BH^* \,-\, HB^* \tag{2b}$$

has a solution $H \in \mathbb{C}^{n\times m}$. Either of these conditions together with the positive definiteness of Σ completely characterizes state covariances of linear dynamical systems driven by white or colored stochastic processes [5], [6]. In particular, when the input is white noise with covariance W, Σ satisfies the algebraic Lyapunov equation

$$A\Sigma \;+\; \Sigma A^* \;=\; -BWB^*. \tag{3}$$

Hence, in this case, H in (2b) is determined by $H = BW/2$.

In general,

$$\begin{aligned} Q \;&:=\; -\left(A\Sigma + \Sigma A^*\right) && \text{(4a)} \\ &=\; BH^* + HB^*. && \text{(4b)} \end{aligned}$$

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is not necessarily positive semi-definite, unless the input is white noise. In fact, for a given Σ a solution H to (2b) allows for the reconstruction of admissible input power spectra [5], [6].

While Q may not necessarily be positive semi-definite, it is not arbitrary. Our motivation in this paper is to complete partially known state statistics with models of low complexity. In order to do this, it is necessary to understand the structural constraints that (4) imposes on Q. In particular, we examine admissible values of the *signature* on Q (that is, the number of positive, negative, and zero eigenvalues). We will show that the number of positive and negative eigenvalues of Q impacts the number of input channels in the state equation (1). We will utilize this information for completion of partially known state statistics.

III. The signature of Q

We now characterize admissible signatures for a Hermitian matrix Q which satisfies (4). There are two sets of constraints arising from (4a) and (4b), respectively. The first one is a standard Lyapunov equation with Hurwitz A and a given Hermitian $\Sigma \succ 0$. The second one, which is a linear equation in B and H, provides the link between the signature of Q and the number of input channels in (1).

A. *Constraints arising from* $Q = -(A\Sigma + \Sigma A^*)$

The unique solution to the Lyapunov equation

$$A\Sigma \,+\, \Sigma A^* \,=\, -Q, \tag{5}$$

with A Hurwitz and Σ, Q Hermitian, is given by

$$\Sigma \,=\, \int_0^\infty \mathrm{e}^{At}\, Q\, \mathrm{e}^{A^* t}\, \mathrm{d}t. \tag{6}$$

Lyapunov theory tells us that if Q is positive definite then Σ is also positive definite. However, the converse is not true; for a given $\Sigma \succ 0$, Q obtained from (5) is not necessarily positive definite. Clearly, Q cannot be negative definite either, otherwise Σ obtained from (6) would be negative semi-definite. This raises the question about admissible signatures of $Q = Q^*$. In what follows, the signature is defined as the triple

$$\mathrm{In}(Q) = (\pi(Q), \nu(Q), \delta(Q))$$

where $\pi(Q), \nu(Q)$, and $\delta(Q)$ denote the number of positive, negative, and zero eigenvalues of Q, respectively.

Several authors have studied constraints on signatures of A, Σ, and Q that are linked through a Lyapunov equation [12]–[14]. Typically, such studies focus on the relationship between the signature of Σ and the eigenvalues of A for a given $Q \succeq 0$. Recently, the authors of [15] have considered the relationship between the signature of Q and eigenvalues of A for $\Sigma \succ 0$. Since one of our objectives is to understand the constraints on the signature of Q arising from the Lyapunov equation (5) with A Hurwitz and $\Sigma \succ 0$ we will make use of a result in [15].

Let $\{\lambda_1, \ldots, \lambda_l\}$ denote the eigenvalues of A, let g_i denote the geometric multiplicity of λ_i, and

$$i(A) \;:=\; \max_{1 \,\leq\, i \,\leq\, l}\, g_i. \tag{7}$$

The following result is a special case of [15, Theorem 2].

Proposition 1: Let A be Hurwitz and let Σ be positive definite. For $Q = -(A\Sigma + \Sigma A^*)$,

$$\pi(Q) \,\geq\, i(A). \tag{8}$$

To explain the nature of the constraint $\pi(Q) \geq i(A)$, we first note that $i(A)$ is the least number of input channels that are needed for system (1) to be controllable. Now consider a decomposition

$$Q \,=\, Q_+ \,-\, Q_-$$

where Q_+, Q_- are positive semi-definite matrices, and accordingly $\Sigma = \Sigma_+ - \Sigma_-$ with Σ_+, Σ_- denoting the solutions of the corresponding Lyapunov equations. Clearly, unless the above constraint (8) holds, Σ_+ cannot be positive definite. Hence, Σ cannot be positive definite either.

Interestingly, there is no constraint on $\nu(Q)$ other than

$$\pi(Q) \,+\, \nu(Q) \,\leq\, n$$

which comes from the dimension of Q.

B. *Constraints arising from* $Q = BH^* + HB^*$

We begin with a basic lemma.

Lemma 1: For a Hermitian matrix Q decomposed as

$$Q \,=\, S \,+\, S^*$$

the following holds

$$\pi(Q) \,\leq\, \mathrm{rank}(S).$$

Proof: The proof is given in the appendix. ∎

Clearly, the same bound applies to $\nu(Q)$, that is,

$$\nu(Q) \,\leq\, \mathrm{rank}(S).$$

The importance of these bounds stems from our interest in decomposing Q into summands of small rank. A decomposition of Q into $S + S^*$ allows us to identify input channels and power spectra by factoring $S = BH^*$. The rank of S coincides with the rank of B, that is, with the number of input channels in the state equation. Thus, it is of interest to determine the minimum rank of S in such a decomposition and this is given in the following proposition.

Proposition 2: For a Hermitian matrix Q having signature $(\pi(Q), \nu(Q), \delta(Q))$,

$$\min\left\{\mathrm{rank}(S) |\; Q = S + S^*\right\} = \max\left\{\pi(Q), \nu(Q)\right\}. \tag{9}$$

Proof: The proof is given in the appendix. ∎

C. *Constraints on the signature of* Q

We now summarize the bounds on the number of positive and negative eigenvalues of the matrix Q defined by (4). By combining Proposition 1 with Lemma 1 we show that these upper bounds are dictated by the number of inputs in the state equation (1).

Proposition 3: Let $\Sigma \succ 0$ denote the steady-state covariance of the state x of a stable linear system (1) with m inputs. If Q satisfies the Lyapunov equation (5), then

$$0 \leq \nu(Q) \leq m \tag{10a}$$

$$i(A) \leq \pi(Q) \leq m. \tag{10b}$$

Proof: Following [5], a state covariance Σ satisfies

$$A\Sigma + \Sigma A^* = -BH^* - HB^*.$$

Setting $S = BH^*$,

$$Q = BH^* + HB^* = S + S^*.$$

From Lemma 1,

$$\max\{\pi(Q), \nu(Q)\} \leq \operatorname{rank}(S) \leq \operatorname{rank}(B) = m.$$

The lower bounds follow from Proposition 1. ■

We note that Proposition 3 does not require $\pi(Q) \geq \nu(Q)$. In fact, for a Hurwitz A with $i(A) = 1$, it is possible to have $\Sigma \succ 0$ with $\nu(Q) \gg \pi(Q) = 1$. To see this, let $Q_1 := -I$ with $\Sigma_1 \prec 0$ the corresponding solution of the Lyapunov equation. Let b be a vector such that (A, b) is controllable, $Q_2 = bb^*$ with $\Sigma_2 \succ 0$ the corresponding steady-state covariance. Choose α large enough so that

$$\Sigma_3 = \alpha\Sigma_2 + \Sigma_1 \succ 0.$$

It follows that $Q_3 := \alpha bb^* - I$ satisfies

$$A\,\Sigma_3 + \Sigma_3\,A^* = -Q_3$$

with $\Sigma_3 \succ 0$. Clearly, the signature of Q_3 is $(1, n-1, 0)$.

Proposition 2, combined with the results of this section, leads to matrix completion problem that we discuss next.

IV. Completion of partially known state covariances

The end goal of the matrix completion problem that we formulate is to obtain information about unknown input excitation in the linear dynamics (1). Typically, in many emerging applications, while the dynamical generator A is known, the observed statistics for the state vector arise from disturbances that are difficult to account for. Hence, herein, we seek an explanation of observed statistics using disturbance models of low complexity. In particular, we want to identify disturbance models that involve the least number of input channels.

A. Covariance matrix completion problem

For colored-in-time forcing u that enters into the state equation through identity matrix, condition (2a) is trivially satisfied. Indeed, any sample covariance Σ can be generated by a linear model (1) with $B = I$. Thus, a disturbance input u that excites all degrees of freedom in the original system can trivially account for the observed statistics and provides no useful information about the underlying physics.

In most physical systems, disturbance can directly excite only a limited number of directions in the state space. For instance, in mechanical systems where inputs represent forces and states represent position and velocity, disturbances can only enter into the velocity equation. This requirement can be formalized by restricting the input to enter into the state equation through a matrix $B \in \mathbb{C}^{n\times m}$ with $m < n$. In this case, the condition for $\Sigma \succ 0$ to be the state covariance of a linear system (A, B) for some stationary zero-mean stochastic input, is equivalent to a solvability of (2b) in terms of a matrix $H \in \mathbb{C}^{n\times m}$ which in turn provides information about the power spectrum of the input.

In our setting, the structure and size of the matrix B in (1) is not known *a priori*. Thus, our objective will be to identify both matrices B and H that reproduce partially available second-order statistics while striking an optimal balance with the complexity of the model. The complexity of the model is reflected by the signature of Q and consequently, through Proposition 2, by the rank of S where $Q = S + S^*$. Since the rank of $S := BH^*$ (cf. (4b)) coincides with the rank of the matrix B, this rank also dictates the number of channels through which disturbance enters into the state equation. Therefore, it is natural to seek an explanation of the data via a choice of a matrix S which has low rank and is consistent with partially available statistics.

The paradigm of low rank solutions to linearly constrained problems has recently received considerable attention due to the confluence of relevant emerging applications and powerful optimization techniques. In fact, low rank approximations of high dimensional data have found use in statistical signal processing, machine learning, and collaborative filtering [9], [16]–[20]. In our problem, additional structural constraints arise from the requirement that partially available second-order statistics are generated by a linear system with a known A-matrix. In what follows, we use these structural constraints to introduce a new paradigm in the study of matrix completion problems.

The rank is a non-convex function of the matrix and the problem of rank minimization is difficult. Recent advances have demonstrated that the nuclear-norm (i.e., the sum of the singular values)

$$\|S\|_* := \sum_{i=1}^{n} \sigma_i(S)$$

represents a good proxy for rank minimization [7]–[11]. Thus, we formulate the following optimization problem.

State-covariance completion problem. Given $A, G_\ell \in \mathbb{C}^{n\times n}$, $g_\ell \in \mathbb{C}$, for $\ell \in \{1, \ldots, N\}$, with A Hurwitz, determine $Q = S + S^*$ where S is obtained by solving the following:

$$\begin{array}{ll} \text{minimize} & \|S\|_* \\ \text{subject to} & A\,\Sigma \,+\, \Sigma\,A^* \,+\, S \,+\, S^* \,=\, 0, \\ & \Sigma \,\succeq\, 0, \\ & \text{trace}\,(G_\ell\,\Sigma) \,=\, g_\ell,\ \ell \,=\, 1,\ldots,N. \end{array} \tag{MC}$$

Here, the matrices A and G_ℓ as well as the scalars g_ℓ are the problem data, while the $n \times n$ matrices S and $\Sigma = \Sigma^*$ are optimization variables. The trace constraints reflect partial second-order known statistics resulting from high-fidelity numerical simulations or experiments on the underlying physical system. Indeed, steady-state correlations of any output $y(t) = Cx(t)$ can be expressed as

$$\begin{array}{rcl} \mathrm{e}_i^* \lim\limits_{t\to\infty} \mathcal{E}\,(y(t)\,y^*(t))\ \mathrm{e}_j & = & \mathrm{e}_i^*\,C\Sigma C^*\,\mathrm{e}_j \\ & = & \text{trace}\,(C^*\mathrm{e}_j\mathrm{e}_i^*C\,\Sigma) \\ & = & \text{trace}\,(G_\ell\,\Sigma) \end{array}$$

where e_i is a vector with 1 as its ith entry. Thus, the g_ℓ's in (MC) represent observed output covariance data suitably restricting the state covariance Σ.

Since both the objective function and the constraints in (MC) are convex, the optimization problem (MC) is convex as well [21]. Furthermore, as shown in [10], [22], nuclear norm minimization can be formulated as a semidefinite program (SDP) and thus, solved efficiently using standard SDP solvers for small size. For large problems, which are typical in many emerging applications, in a companion paper [23], we develop efficient algorithms. These are based on the alternating direction method of multipliers, a state-of-the-art technique for solving large-scale and distributed optimization problems [24].

Remark 2: As shown in Proposition 2, minimizing the rank of S is equivalent to minimizing $\max\{\pi(Q),\nu(Q)\}$. Given Q with signature $(\pi(Q),\nu(Q),\delta(Q))$, there exist matrices $Q_+ \succeq 0$ and $Q_- \succeq 0$ with $Q = Q_+ - Q_-$ such that $\text{rank}(Q_+) = \pi(Q)$ and $\text{rank}(Q_-) = \nu(Q)$. Furthemore, any such decomposition of Q satisfies $\text{rank}(Q_+) \geq \pi(Q)$ and $\text{rank}(Q_-) \geq \nu(Q)$. Thus, instead of (MC), an alternative convex optimization problem aimed at minimizing $\max\{\pi(Q),\nu(Q)\}$ is given by

$$\begin{array}{ll} \text{minimize} & \max\{\text{trace}(Q_+),\, \text{trace}(Q_-)\} \\ \text{subject to} & A\,\Sigma \,+\, \Sigma\,A^* \,+\, Q_+ \,-\, Q_- \,=\, 0 \\ & \Sigma \,\succeq\, 0,\ Q_+ \succeq 0,\ Q_- \succeq 0 \\ & \text{trace}\,(G_\ell\,\Sigma) \,=\, g_\ell,\ \ell \,=\, 1,\ldots,N. \end{array} \tag{MC1}$$

B. Factorization of Q into $BH^* + HB^*$

If $Q = S + S^*$ is obtained by solving problem (MC), singular value decomposition of S can be used to factor it into $S = BH^*$. If instead, $Q = Q_+ - Q_-$ is obtained by solving problem (MC1), we next demonstrate how the proof of Proposition 2 can be used to decompose the matrix Q into $BH^* + HB^*$ with $S = BH^*$ of minimum rank. Given Q with signature $(\pi(Q),\nu(Q),\delta(Q))$, we can choose an invertible matrix T to bring Q into the following form

$$\hat{Q} \;:=\; TQT^* \;=\; 2\begin{bmatrix} I_\pi & 0 & 0 \\ 0 & -I_\nu & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

where I_π and I_ν are identity matrices of dimension $\pi(Q)$ and $\nu(Q)$ [25, pages 218–223]. We first present factorization of Q for $\pi(Q) \leq \nu(Q)$. With

$$\hat{S} \;=\; \begin{bmatrix} I_\pi & -I_\pi & 0 & 0 \\ I_\pi & -I_\pi & 0 & 0 \\ 0 & 0 & -I_{\nu-\pi} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

we clearly have $\hat{Q} = \hat{S} + \hat{S}^*$. Furthermore, $\hat{S}$ can be written as $\hat{S} = \hat{B}\hat{H}^*$, where

$$\hat{B} \;=\; \begin{bmatrix} I_\pi & 0 \\ I_\pi & 0 \\ 0 & I_{\nu-\pi} \\ 0 & 0 \end{bmatrix},\quad \hat{H} \;=\; \begin{bmatrix} I_\pi & 0 \\ -I_\pi & 0 \\ 0 & -I_{\nu-\pi} \\ 0 & 0 \end{bmatrix}.$$

Finally, the matrices B and H are determined by $B = T^{-1}\hat{B}$ and $H = T^{-1}\hat{H}$.

For $\pi(Q) > \nu(Q)$, Q can be decomposed into $BH^* + HB^*$ with $B = T^{-1}\hat{B}$, $H = T^{-1}\hat{H}$, and

$$\hat{B} \;=\; \begin{bmatrix} I_{\pi-\nu} & 0 \\ 0 & I_\nu \\ 0 & I_\nu \\ 0 & 0 \end{bmatrix},\quad \hat{H} \;=\; \begin{bmatrix} I_{\pi-\nu} & 0 \\ 0 & I_\nu \\ 0 & -I_\nu \\ 0 & 0 \end{bmatrix}.$$

Note that both B and H are full column rank matrices.

C. Input power spectra

Starting from a set of values for the two matrices B and H, which we can obtain using our earlier scheme, we can now determine power spectra for the input to the linear system (1) that are consistent with the steady-state covariance Σ. Indeed, given any $\Omega \succ 0$, the following (matrix-valued) power spectral density

$$\Pi_{uu}(\omega) \;:=\; \Psi(j\omega)\,\Omega\,\Psi(j\omega)^*,$$

where

$$\begin{array}{rcl} \Psi(j\omega) & := & I + C_1(j\omega I - A_1)^{-1}B \\ C_1 & := & -\dfrac{1}{2}\,\Omega B^*\Sigma^{-1} + H^*\Sigma^{-1} \\ A_1 & := & A + BC_1, \end{array}$$

can serve as the power spectrum of an input process u which is consistent with the observed statistics. To verify this, note that

$$(sI - A)^{-1}B\,\Psi(s) \;=\; (sI - A_1)^{-1}B.$$

This step amounts to removing unobservable modes in the series connection of the two systems with transfer functions $(sI - A)^{-1}B$ and $\Psi(s)$. Then, $A_1\Sigma + \Sigma A_1^* + B\Omega B^* = 0$ can be readily verified, which proves our claim.

V. EXAMPLE

Consider a mass-spring-damper (MSD) system with 10 masses subject to colored disturbance. The disturbance is generated by white noise passing through a shaping input filter. The cascade connection of the MSD dynamics together with the input filter is as follows:

$$\begin{array}{rrcl} \text{input filter:} & \dot{z} & = & A_f z + B_f d \\ & u & = & C_f z + D_f d \\ \text{MSD system:} & \dot{x} & = & Ax \,+\, Bu. \end{array}$$

Here, $d(t)$ is zero-mean unit-variance white noise,

$$A \;=\; \left[\begin{array}{cc} O & I \\ -K & -2K \end{array}\right], \quad B \;=\; \left[\begin{array}{c} O \\ I \end{array}\right],$$

where O, I represent the zero and identity matrices, respectively, and K is a symmetric tridiagonal Toeplitz matrix with 2 on its main diagonal and -1 on the first upper and lower sub-diagonals.

The state covariance Σ of the cascade system is the solution of the Lyapunov equation

$$\mathcal{A}\Sigma \,+\, \Sigma\mathcal{A}^* \,+\, \mathcal{B}\mathcal{B}^* \;=\; 0, \tag{13}$$

where

$$\mathcal{A} \;=\; \left[\begin{array}{cc} A & BC_f \\ O & A_f \end{array}\right], \quad \mathcal{B} \;=\; \left[\begin{array}{c} BD_f \\ B_f \end{array}\right].$$

We partition

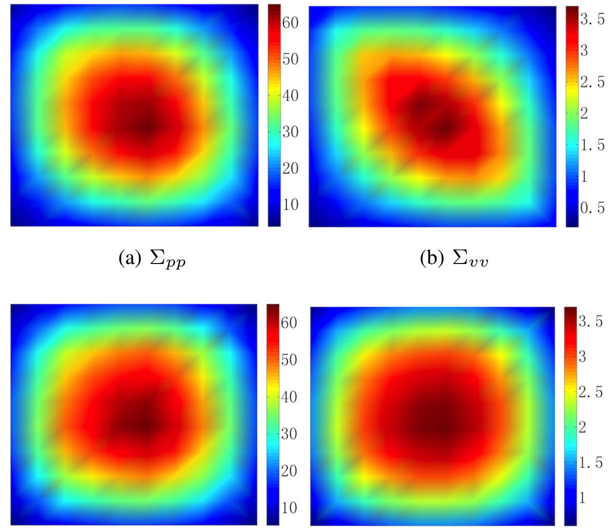
$$\Sigma \;=\; \left[\begin{array}{cc} \Sigma_{xx} & \Sigma_{xz} \\ \Sigma_{zx} & \Sigma_{zz} \end{array}\right],$$

and further partition the sub-covariance of the MSD system

$$\Sigma_{xx} \;=\; \left[\begin{array}{cc} \Sigma_{pp} & \Sigma_{pv} \\ \Sigma_{vp} & \Sigma_{vv} \end{array}\right],$$

in order to highlight the covariances Σ_{pp} and Σ_{vv} of the position and velocity components, respectively. We apply the earlier methodology to complete Σ_{xx} using only knowledge of the diagonal elements of Σ_{pp} and Σ_{vv}.

In this example, we set z to have the same number of components as x and select A_f to be a diagonal matrix with random negative entries. Furthermore, for $B_f \;=\; C_f \;=\; D_f \;=\; I$, we solve the optimization problem (MC1) and obtain $Q = S + S^*$ with 2 positive eigenvalues (11.3824, 0.6614), 2 negative eigenvalues $(-12.0437, -0.0001)$, and 16 eigenvalues at 0. Thus, we use 2 colored-in-time inputs to account for the given diagonal partial state statistics. In Figs. 1a and 1b we display the color-coded Σ_{pp} and Σ_{vv} obtained by solving the Lyapunov equation (13). These represent the "ground truth", since the known disturbance model is used. In Figs. 1c and 1d we display the reconstructed $\hat{\Sigma}_{pp}$ and $\hat{\Sigma}_{vv}$, obtained by the optimization problem (MC1) using only the diagonal entries of Σ_{pp} and Σ_{vv}. We observe that $\hat{\Sigma}_{pp}$ and $\hat{\Sigma}_{vv}$ are fairly good approximations of Σ_{pp} and Σ_{vv}.



(a) Σ_{pp} (b) Σ_{vv}

(c) $\hat{\Sigma}_{pp}$ (d) $\hat{\Sigma}_{vv}$

Fig. 1: First row: submatrices Σ_{pp} and Σ_{vv} of the state covariance of the MSD system. Second row: matrix completions $\hat{\Sigma}_{pp}$ and $\hat{\Sigma}_{vv}$ based on (MC1).

VI. CONCLUDING REMARKS

We are interested in explaining partially known second-order statistics of a linear system by the least possible number of input disturbances. This problem arises from the need for model-based analysis and design techniques in control of turbulent fluid flows. In such an application, the linearized NS equations provide the generator for the dynamics whereas the nature and directionality of disturbances that can account for the observed statistics are largely unknown. The disturbance model, both in terms of directionality as well as spectral content, is sought as a solution to an optimization problem. Analysis of the signature of relevant optimization parameters provides insight into structure of the disturbance subspace and motivates alternative formulations. The dimensionality of the disturbance vector introduces a rank constraint which is relaxed using the nuclear norm proxy.

APPENDIX

Proof of Lemma 1

Without loss of generality, let us consider Q of the following form

$$Q = 2\left[\begin{array}{ccc} I_\pi & 0 & 0 \\ 0 & -I_\nu & 0 \\ 0 & 0 & 0 \end{array}\right]$$

as in (11). Given any S that satisfies $Q = S + S^*$ we can decompose it into

$$S = M + N$$

with M Hermitian and N skew-Hermitian. It is easy to see that

$$M = \frac{Q}{2} = \begin{bmatrix} I_\pi & 0 & 0 \\ 0 & -I_\nu & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If we write N as

$$N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix}$$

then

$$S = \begin{bmatrix} I_\pi + N_{11} & N_{12} & N_{13} \\ N_{21} & -I_\nu + N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix}.$$

Clearly,

$$\mathrm{rank}(S) \geq \mathrm{rank}(I_\pi + N_{11}).$$

Since N_{11} is a skew-Hermitian matrix, all its eigenvalues are on the imaginary axis. This implies that all the eigenvalues of $I_\pi + N_{11}$ have real part 1 and therefore $I_\pi + N_{11}$ is a full rank matrix. Hence, we have

$$\mathrm{rank}(S) \geq \mathrm{rank}(I_\pi + N_{11}) = \pi(Q)$$

which completes the proof.

Proof of Proposition 2

The inequality

$$\min\{\mathrm{rank}(S) |\ Q = S + S^*\} \geq \max\{\pi(Q), \nu(Q)\}$$

follows from Lemma 1. To establish the proposition we need to show that the bounds are tight, i.e.,

$$\min\{\mathrm{rank}(S) |\ Q = S + S^*\} \leq \max\{\pi(Q), \nu(Q)\}.$$

Given Q in (11), for $\pi(Q) \leq \nu(Q)$, Q can be written as

$$Q = 2 \begin{bmatrix} I_\pi & 0 & 0 & 0 \\ 0 & -I_\pi & 0 & 0 \\ 0 & 0 & -I_{\nu-\pi} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By selecting S in the form (12) we conclude that

$$\begin{aligned} \mathrm{rank}(S) &= \mathrm{rank}(\begin{bmatrix} I_\pi & -I_\pi \\ I_\pi & -I_\pi \end{bmatrix}) + \mathrm{rank}(-I_{\nu-\pi}) \\ &= \pi(Q) + \nu(Q) - \pi(Q) = \nu(Q). \end{aligned}$$

Therefore

$$\min\{\mathrm{rank}(S) |\ Q = S + S^*\} \leq \nu(Q).$$

Similarly, for the case $\pi(Q) > \nu(Q)$,

$$\min\{\mathrm{rank}(S) |\ Q = S + S^*\} \leq \pi(Q).$$

Hence,

$$\min\{\mathrm{rank}(S) |\ Q = S + S^*\} \leq \max\{\pi(Q), \nu(Q)\}$$

which completes the proof.

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