Nonadaptive group testing with random set of defectives via constant-weight codes

Arya Mazumdar Member, IEEE

Abstract

In a group testing scheme, set of tests are designed to identify a small number $t$ of defective items that are present among a large number $N$ of items. Each test takes as input a group of items and produces a binary output indicating whether any defective item is present in the group. In a non-adaptive scheme the tests have to be designed in one-shot. In this setting, designing a testing scheme is equivalent to the construction of a disjunct matrix, an $M \times N$ binary matrix where the union of supports of any $t$ columns does not contain the support of any other column. In principle, one wants to have such a matrix with minimum possible number $M$ of rows.

In this paper we consider the scenario where defective items are random and follow simple probability distributions. In particular we consider the cases where 1) each item can be defective independently with probability $\frac{t}{N}$ and 2) each $t$-set of items can be defective with uniform probability. In both cases our aim is to design a testing matrix that successfully identifies the set of defectives with high probability. Both of these models have been studied in the literature before and it is known that $O(t \log N)$ tests are necessary as well as sufficient (via random coding) in both cases.

Our main focus is explicit deterministic construction of the test matrices amenable to above scenarios. One of the most popular ways of constructing test matrices relies on constant-weight error-correcting codes and their minimum distance. In particular, it is known that codes result in test matrices with $O(t^2 \log N)$ rows that identify any $t$ defectives. We go beyond the minimum distance analysis and connect the average distance of a constant weight code to the parameters of the resulting test matrix. Indeed, we show how distance, a pairwise property of the columns of the matrix, translates to a $(t+1)$-wise property.

The author is with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455, email: arya@umn.edu. Part of this work was presented in the International Symposium on Algorithms and Computation, 2012 [31].
of the columns. With our relaxed requirements, we show that using an explicit constant-weight code we may achieve a number of tests equal to $O(t \log^2 N / \log t)$ for the first case and $O(t^{3/2} \log^{3/2} N / \log t)$ for the second case. While still away by a factor of $\log N / \log t$ and $\sqrt{t \log N / \log t}$ respectively from the optimal number of tests, one may note that our constructions are deterministic and the main contribution lies in relating the group testing properties to parameters of constant-weight codes.

I. INTRODUCTION

Combinatorial search is an old and well-studied problem. In the most general form it is assumed that there is a set of $N$ elements among which at most $t$ are defective. This set of defective items is called the defective set or configuration. To find the defective set, one might test all the elements individually for defects, requiring $N$ tests. Intuitively, that would be a waste of resource if $t \ll N$. On the other hand, to identify the defective configuration it is required to ask at least $\log \sum_{i=0}^{t} \binom{N}{i} \approx t \log \frac{N}{t}$ yes-no questions. The main objective is to identify the defective configuration with a number of tests that is as close to this minimum as possible.

In the group testing problem, a group of elements are tested together and if this particular group contains any defective element the test result is positive. Based on the test results of this kind one identifies (with an efficient algorithm) the defective set with minimum possible number of tests. The schemes (grouping of elements) can be adaptive, where the design of one test may depend on the results of preceding tests. For a comprehensive survey of adaptive group testing schemes we refer the reader to [13].

In this paper we are interested in non-adaptive group testing schemes: here all the tests are designed together. If the number of designed tests is $M$, then a non-adaptive group testing scheme is equivalent to the design of a binary test matrix of size $M \times N$ where the $(i,j)$th entry is 1 if the $i$th test includes the $j$th element; it is 0 otherwise. As the test results, we see the Boolean OR of the columns corresponding to the defective entries.

Extensive research has been performed to find out the minimum number of required tests $M$ in terms of the number of elements $N$ and the maximum number of defective elements $t$. The best known lower bound says that it is necessary to have $M = \Omega(\frac{t^2}{\log t} \log N)$ tests [14], [17]. The existence of non-adaptive group testing schemes with $M = O(t^2 \log N)$ is also known for quite some time [13], [23]. On the other hand, for the adaptive setting, schemes have been constructed with as small as $O(t \log N)$ tests, optimal up to a constant factor [13], [22].

December 11, 2014  DRAFT
In the literature, many relaxed versions of the group testing problem have been studied as well. For example, in [18], [42] recovery of a list of items containing the true defectives is suggested (list-decoding superimposed codes). This notion was revisited in [8], [25] as list-disjunct matrix and in [21], where it was assumed that recovering a large fraction of defective elements is sufficient. There are also information-theoretic models for the group testing problem where the test results can be noisy [2] (also see [5], [7]). In other versions of the group testing problem, a test may carry more than one bit of information [4], [24], or the test results are threshold-based (see [9] and references therein). Algorithmic aspects of the recovery schemes have been studied in several papers. For example, papers [25] and [36] provide efficient recovery algorithms for non-adaptive group testing. There are possibly many random models that can be defined on the defectives (see, e.g. [19]).

Here as well, we consider two relaxed versions of the group testing problem – we want recovery to be successful with high probability assuming uniform distributions of the defective items. In the first scenario, each of the N items can be defective with probability \( tN \). This model of defectives, called Model 1 below, is as old as the group testing problem [12] and was rigorously defined in [39]. We provide explicit construction of test matrices with \( O(t \log^2 N/\log t) \) tests for this situation. In the second scenario, we want the recovery to be successful for a very large faction of all possible \( t \)-sets as defective configurations. This scenario, called Model 2 below, was considered under the name of weakly separated design in [30], [45] and [28]. It is known (see, [45]) that, with this relaxation it might be possible to reduce the number of tests to be proportional to \( t \log N \). However this result is not constructive\(^1\). Here we provide explicit construction of test matrices with \( O((t \log N)^{3/2}/\log t) \) tests.

Non-adaptive group testing has found applications in multiple different areas, such as, multi-user communication [3], [43], DNA screening [35], pattern finding [27] etc. It can be observed that in many of these applications it would have been still useful to have a scheme that identifies almost all different defective configurations if not all possible defective configurations. The above relaxations form a parallel of similar works in compressive sensing (see, [6], [32]) where recovery of almost all sparse signals from a generic random model is considered.

A construction of group testing schemes from error-correcting code matrices and using code con-

\(^1\)Recently, in [20], a construction of weakly-separated design is presented that requires \( O(t \cdot \text{polylog}(N)) \) tests. However the focus of our paper is a construction from error-correcting codes.
Catenation appeared in the seminal paper by Kautz and Singleton [26]. Code concatenation is a way to construct binary codes from codes over a larger alphabet [29]. In [26], the authors concatenate a q-ary (q > 2) Reed-Solomon code with a unit-weight code to use the resulting codewords as the columns of the testing matrix. Recently in [37], an explicit construction of a scheme with $M = O(t^2 \log N)$ tests is provided. The construction of [37] is based on the idea of [26]: instead of the Reed-Solomon code, they take a low-rate code that achieves the Gilbert-Varshamov bound of coding theory [29], [38]. Papers, such as [16], [44], also consider construction of non-adaptive group testing schemes.

In this paper we show that the explicit construction of [37] based on error-correcting codes works for both Model 1 and Model 2 and results in numbers of tests claimed above.

A. Results and organization

The constructions of [26], [37] and many others are based on constant-weight error-correcting codes, a set of binary vectors of same Hamming weight (number of ones). The group-testing recovery property relies on the pairwise minimum distance between the vectors of the code [26]. In this work, we go beyond this minimum distance analysis and relate the group-testing parameters to the average distance of the constant-weight code. This allows us to connect the group testing matrices designed for random models of defectives to error-correcting codes in a general way. Previously the connection between distances of the code and weakly separated designs was only known for the very specific family of maximum distance separable codes [28], where much more information than the average distance is evident.

Based on the newfound connection, for both Model 1 and Model 2, we construct explicit (constructible deterministically in polynomial time) families of non-adaptive group testing schemes. For Model 1, our nonadaptive scheme can identify the set of defectives exactly with probability $1 - \epsilon$. The total number of requires tests are $6t \ln N \ln \frac{N}{\epsilon} / \log t$. For Model 2, we show that our scheme scheme identifies $(1 - \epsilon)$ proportion of all possible defective sets of size $t$ using $4t^{3/2} \log N \sqrt{\log \frac{N}{\epsilon}} / \log t$ tests. It can be seen that, with the relaxation in requirement, the number of tests is brought down to be proportional to $t$ and $t^{3/2}$, respectively, from $t^2$. Note that, in this paper we assume $\log N = o(t)$.

Our construction technique is same as the scheme of [26], [37], however with a finer analysis relying on the distance properties of a linear code we are able to achieve more. One of the main contribution is to show a general way to establish a property for almost all t-tuples of elements from a set based on the
mean pairwise statistics of the set.

In Section II, we provide the necessary definitions and preliminaries. The relation of group testing parameters of Model 1 with constant-weight codes is provided in Section III. In Section IV we establish the connection between the parameters of a weakly separated design and the average distance of a constant-weight code. In Section V we discuss our construction scheme that works for both of Models 1 and 2.

II. BASIC DEFINITIONS AND PROPERTIES

A vector is denoted by bold lowercase letters, such as \( \mathbf{x} \), and the \( i \)th entry of the vector \( \mathbf{x} \) is denoted by \( x_i \). The Hamming distance between two vectors is denoted by \( d_H(\cdot, \cdot) \). The support of a vector \( \mathbf{x} \) is the set of coordinates where the vector has nonzero entries. It is denoted by \( \text{supp}(\mathbf{x}) \). We use the usual set terminology, where a set \( A \) contains \( B \) if \( B \subseteq A \). Also, below \([n]\) denotes \( \{1, 2, \ldots, n\}\).

A. Disjunct matrices

Definition 1: An \( M \times N \) binary matrix \( \mathbf{A} \) is called \( t \)-disjunct if the support of any column is not contained in the union of the supports of any other \( t \) columns.

It is not very difficult to see that a \( t \)-disjunct matrix gives a group testing scheme that identifies any defective set up to size \( t \). On the other hand any group testing scheme that identifies any defective set up to size \( t \) must be a \((t-1)\)-disjunct matrix [13]. The definition of disjunct matrix can be restated as follows: a matrix is \( t \)-disjunct if any \( t+1 \) columns indexed by \( i_1, \ldots, i_{t+1} \) of the matrix form a sub matrix which must have a row that has exactly one 1 in the \( i_j \)th position and zeros in the other positions, for \( j = 1, \ldots, t+1 \).

To a great advantage, disjunct matrices allow for a simple identification algorithm that runs in time \( O(Nt) \).

B. Disjunct decoding

Given the test results \( \mathbf{y} \in \{0, 1\}^M \), we use the following recovery algorithm to find the defectives. Suppose, \( \mathbf{A} \) is the test matrix and \( \mathbf{a}_j \in \{0, 1\}^N, j = 1, \ldots, M \) denotes the \( j \)th row of \( \mathbf{A} \). The recovery algorithm simply outputs

\[
[N] \setminus \bigcup_{j:y_j=0} \text{supp}(\mathbf{a}_j)
\]
as the set of defectives.

Note that, irrespective of the testing matrix, this algorithm will always output a set that contains all the defective elements. Moreover, if the testing matrix is disjunct, then the output is exactly equal to the set of defectives. We have the following simple proposition.

**Proposition 1:** Suppose, the set of defectives is \( S \subseteq [N] \). Let \( a^{(k)} \) denote the kth column of the test matrix \( A \). Then the disjunct decoding algorithm recovers the defectives exactly if \( \bigcup_{j \in S} \text{supp}(a^{(j)}) \) does not contain the support of \( a^{(i)} \) for all \( i \in [N] \setminus S \).

### C. Almost disjunct matrices

Below we define a relaxed form of disjunct matrices. This definition appeared very closely in [30], [45] and exactly in [28].

**Definition 2:** For any \( \epsilon > 0 \), an \( M \times N \) matrix \( A \) is called \((t, \epsilon)\)-disjunct if the set of \( t \)-tuple of columns (of size \( \binom{N}{t} \)) has a subset \( B \) of size at least \( (1 - \epsilon) \binom{N}{t} \) with the following property: for all \( J \in B \), \( \bigcup_{\kappa \in J} \text{supp}(\kappa) \) does not contain support of any column \( \nu \not\in J \).

In other words, the union of supports of a randomly and uniformly chosen set of \( t \) columns from a \((t, \epsilon)\)-disjunct matrix does not contain the support of any other column with probability at least \( 1 - \epsilon \).

It is clear that for \( \epsilon = 0 \), the \((t, \epsilon)\)-disjunct matrices are same as \( t \)-disjunct matrices.

It is easy to see the following fact.

**Proposition 2 (Model 2):** A \((t, \epsilon)\)-disjunct matrix gives a group testing scheme that can identify all but at most a fraction \( \epsilon > 0 \) of all possible defective configurations of size \( t \).

### D. Constant-weight codes

A binary \((M, N, d)\) code \( \mathcal{C} \) is a set of size \( N \) consisting of \( \{0,1\} \)-vectors of length \( M \). Here \( d \) is the largest integer such that any two vectors (codewords) of \( \mathcal{C} \) are at least Hamming distance \( d \) apart. \( d \) is called the minimum distance (or distance) of \( \mathcal{C} \). If all the codewords of \( \mathcal{C} \) have Hamming weight \( w \), then it is called a constant-weight code. In that case we write \( \mathcal{C} \) is an \((M, N, d, w)\)-constant-weight binary code.

Constant-weight codes can give constructions of group testing schemes. One just arranges the codewords as the columns of the test matrix. Kautz and Singleton proved the following in [26].
Proposition 3: An \((M,N,d,w)\)-constant-weight binary code provides a \(t\)-disjunct matrix where, \(t = \left\lfloor \frac{w-1}{w-d/2} \right\rfloor \).

Proof: The intersection of supports of any two columns has size at most \(w - d/2\). Hence if \(w > t(w - d/2)\), support of any column will not be contained in the union of supports of any \(t\) other columns.

Extensions of Prop. 3 are our main results. To do that we need to define the average distance \(D\) of a code \(C\):

\[
D(C) = \frac{1}{|C|} \min_{x \in C} \sum_{y \in C} d_H(x, y).
\]

Here \(d_H(x, y)\) denotes the Hamming distance between \(x\) and \(y\).

III. MODEL 1: INDEPENDENT DEFECTIVES - TEST MATRICES FROM CONSTANT-WEIGHT CODES

In this section, we consider the independent failure model (Model 1) and show how the minimum and average distances of a constant-weight binary code contribute to a nonadaptive group testing scheme. Recall, in this model we assume that among \(N\) items, each is defective with a probability \(tN\). The main result of this section is the following theorem.

**Theorem 1 (Model 1):** Suppose, we have a constant-weight binary code \(C\) of size \(N\), minimum distance \(d\) and average distance \(D\) such that every codeword has length \(M\) and weight \(w\). The test matrix obtained from the code exactly identifies all the defective items with probability at least \(1 - \epsilon\) if

\[
w - \frac{d}{2} \leq \frac{3(w - t(w - D/2))^2}{2\left(2t(w - D/2 + w)\right) \ln \frac{N}{\epsilon}}. \tag{1}
\]

We will need the help of the following lemma to prove the theorem. Note, from Prop. 1, the disjunct-recovery algorithm will be successful if the union of supports of the columns corresponding to the defectives does not contain the support of any other columns. Suppose the testing matrix is constructed from an \((M,N,d,w)\)-constant-weight code \(C\) (each column is a codeword). Let

\[
C = \{c_1, c_2, \ldots, c_2\}.
\]

Moreover, assume \(X_j \in \{0, 1\}\) is the indicator Bernoulli(t/N) random variable that denotes whether the \(j\)th element is defective or not.
Lemma 1: Suppose, for all \( i \in [N] \), we have
\[
\sum_{j=1, j \neq i}^{N} X_j \left( w - \frac{d_H(c_i, c_j)}{2} \right) < w.
\]
Then the disjunct-recovery algorithm will exactly identify the defective elements.

Proof: The lemma directly follows from Prop. 1 and the fact that for any \( i, j \), \( w - \frac{d_H(c_i, c_j)}{2} \) is nonnegative. Suppose \( S \subseteq [N] \) be the random set of defectives. The disjunct-recovery algorithm will be successful when for all \( i \in [N] \setminus S \),
\[
\sum_{j \in S} \left( w - \frac{d_H(c_i, c_j)}{2} \right) < w.
\]
Hence the condition of the lemma is sufficient for success.

Now we are ready to prove Thm. 1.

Proof of Thm. 1: First of all, note that,
\[
\Pr \left( \exists i \in [N] : \sum_{j=1, j \neq i}^{N} X_j \left( w - \frac{d_H(c_i, c_j)}{2} \right) \geq w \right) \leq \sum_i \Pr \left( \sum_{j=1, j \neq i}^{N} X_j \left( w - \frac{d_H(c_i, c_j)}{2} \right) \geq w \right).
\]
For a fixed \( i \), we would want to upper bound the probability above in the right hand side under the summation. Assume, \( w - \frac{d_H(c_i, c_j)}{2} = a_j \). Notice, \( a_j X_j - \mathbb{E}(a_j X_j) \leq a_j (1 - t/N) \leq (1 - t/N)(w - d/2) \) and \( \sum_{j \neq i} a_j X_j = a_j X_j + a_j t/N ) = \frac{1}{N} \left( 1 - \frac{1}{N} \right) \sum_{j \neq i} a_j^2 \).

Now, we use a classical concentration inequality of [34, Thm. 2.7], to have,
\[
\Pr \left( \sum_{j=1, j \neq i}^{N} X_j \left( w - \frac{d_H(c_i, c_j)}{2} \right) \geq w \right) = \Pr \left( \sum_{j=1, j \neq i}^{N} (X_j - t/N)a_j \geq w - \frac{t}{N} \sum_{j \neq i} a_j \right)
\leq \Pr \left( \sum_{j=1}^{N} (X_j - t/N)a_j \geq w - \frac{t}{N} \sum_{j=1}^{N} a_j \right)
\leq \exp \left( - \frac{\left( w - \frac{t}{N} \sum_{j=1}^{N} a_j \right)^2}{2 \left( \frac{t}{N} \left( 1 - \frac{1}{N} \right) \sum_{j=1}^{N} a_j^2 + \frac{1}{3} \left( 1 - \frac{t}{N} \right) \left( w - \frac{d}{2} \right) \left( w - \frac{t}{N} \sum_{j=1}^{N} a_j \right) \right)} \right)
\leq \exp \left( - \frac{\left( w - \frac{1}{N} \sum_{j=1}^{N} a_j \right)^2}{2 \left( \frac{1}{N} \sum_{j=1}^{N} a_j^2 + \frac{1}{3} \left( w - \frac{d}{2} \right) \left( w - \frac{1}{N} \sum_{j=1}^{N} a_j \right) \right)} \right)
\leq \exp \left( - \frac{\left( w - \frac{1}{N} \sum_{j=1}^{N} a_j \right)^2}{2 \left( \frac{1}{N} \sum_{j=1}^{N} a_j \left( w - \frac{d}{2} \right) + \frac{1}{3} \left( w - \frac{d}{2} \right) \left( w - \frac{1}{N} \sum_{j=1}^{N} a_j \right) \right)} \right)
\leq \exp \left( - \frac{\left( w - \frac{1}{N} \sum_{j=1}^{N} a_j \right)^2}{2 \left( \frac{1}{N} \sum_{j=1}^{N} a_j \left( w - \frac{d}{2} \right) + \frac{1}{3} \left( w - \frac{d}{2} \right) \left( w - \frac{1}{N} \sum_{j=1}^{N} a_j \right) \right)} \right)
\]
\[ \leq \exp \left( -\frac{3 \left( w - \frac{t}{N} \sum_j a_j \right)^2}{2 \left( w - \frac{d}{2} \right) \left( \frac{2t}{N} \sum_j a_j + w \right)} \right) \]

\[ \leq \exp \left( -\frac{3 \left( w - t(w - D/2) \right)^2}{2 \left( w - \frac{d}{2} \right) \left( 2t(w - D/2) + w \right)} \right), \]

where the last inequality is true because the exponent above is an increasing function of \( \sum_j a_j \) and \( \frac{1}{N} \sum_j a_j = w - \frac{1}{2N} \sum_j d_{ij}(c_i, c_j) \leq w - \frac{D}{2} \). Now using union bound, we deduce that the test matrix will successfully identify the defective elements exactly with probability \( 1 - \epsilon \) if

\[ \frac{3 \left( w - t(w - D/2) \right)^2}{2 \left( w - \frac{d}{2} \right) \left( 2t(w - D/2) + w \right)} \geq \ln \frac{N}{\epsilon}, \]

which proves the theorem.

Similar result can be obtained for Model 2. However, because of the dependence among the random choice of defectives we need to use Martingale concentration inequalities.

IV. MODEL 2: \((t, \epsilon)\)-DISJUNCT MATRICES FROM CONSTANT-WEIGHT CODES

Our main result of this section is the following.

Theorem 2: Suppose, we have a constant-weight binary code \( C \) of size \( N \), minimum distance \( d \) and average distance \( D \) such that every codeword has length \( M \) and weight \( w \). The test matrix obtained from the code is \((t, \epsilon)\)-disjunct for the largest \( t \) such that,

\[ \alpha \sqrt{t \ln \frac{N - t}{\epsilon}} \leq \frac{w - t(w - D/2)}{w - d/2} \]

holds. Here \( \alpha \) is any absolute constant greater than or equal to \( \sqrt{2}(1 + t/N) \).

One can compare the results of Prop. 3 and Theorem 2 to see the improvement achieved as we relax the definition of disjunct matrices. This will lead to the final improvement on the parameters of Porat-Rothschild construction [37], as we will see in Section V.

A. Proof of Theorem 2

This section is dedicated to the proof of Theorem 2. Suppose, we have a constant-weight binary code \( C \) of size \( N \) and minimum distance \( d \) such that every codeword has length \( M \) and weight \( w \). Let the
average distance of the code be \( D \). Note that this code is fixed: we will prove a property of this code by probabilistic method.

Let us now choose \( t \) codewords randomly and uniformly from all possible \( \binom{N}{t} \) choices. Let the randomly
chosen codewords are \( \{ c_1, c_2, \ldots, c_t \} \). In what follows, we adapt the proof of Prop. 3 in a probabilistic setting.

Assume we call the random set of defectives as \( S \). For \( l \in [N] \setminus S \), define the random variables
\[
Z_l = \sum_{j=1}^{t} \left( w - \frac{d_H(c_l, c_j)}{2} \right).
\]
Clearly, \( Z_l \) is the maximum possible size of the portion of the support of \( c_l \) that is common to at least one of \( c_j, j = 1, \ldots, t \). Note that the size of support of \( c_l \) is \( w \). Hence, as we have seen in the proof of Prop. 3, if \( Z_l \) is less than \( w \) for all \( l \) that are not part of the defective set, then the disjunct decoding algorithm will be successful. Therefore, we aim to find the probability \( \Pr(\exists l \in [N] \setminus S : Z_l \geq w) \) and show it to be bounded above by \( \epsilon \) under the condition of the theorem.

We might take \( l \) uniformly distributed in \( [N] \setminus S \). We see that,
\[
\Pr(\exists l \in [N] \setminus S : Z_l \geq w) \leq (N-t) \Pr(Z_l \geq w).
\]
In the following, we will find an upper bound on \( \Pr(Z_l \geq w) \).

Define,
\[
Z_i = \mathbb{E}\left( \sum_{j=1}^{t} \left( w - \frac{d_H(c_l, c_j)}{2} \right) \mid d_H(c_l, c_k), k = 1, 2, 3, \ldots, i \right).
\]
Clearly, \( Z_0 = \mathbb{E}\left( \sum_{j=1}^{t} \left( w - \frac{d_H(c_l, c_j)}{2} \right) \right) \), and \( Z_t = \sum_{j=1}^{t} \left( w - \frac{d_H(c_l, c_j)}{2} \right) = Z_l \).

Now,
\[
Z_0 = \mathbb{E}\left( \sum_{j=1}^{t} \left( w - \frac{d_H(c_l, c_j)}{2} \right) \right) = tw - \frac{1}{2} \mathbb{E} \sum_{j=1}^{t} d_H(c_l, c_j),
\]
where the expectation is over the randomly and uniformly chosen \((t+1)\) codewords from all possible \( \binom{N}{t+1} \) choices. Note,
\[
\mathbb{E} \sum_{j=1}^{t} d_H(c_l, c_j) = \sum_{i_1 < i_2 < \cdots < i_{t+1}} \frac{1}{(N+1)} \sum_{j=2}^{t+1} d_H(c_{i_1}, c_{i_j})
\]
\[
= \frac{1}{(t+1)!} \sum_{1 \leq l \neq m \leq t+1} \sum_{1 \leq j \leq t} d_H(c_{l_i}, c_{i_j}) = \frac{1}{N(N-1)} \sum_{j=2}^{t+1} \sum_{i_1=1}^{N} \sum_{i_j \neq i_1}^{N} d_H(c_{i_1}, c_{i_j})
\]
\[ = \sum_{j=2}^{t+1} \mathbb{E}d_H(c_{i1}, c_{ij}) \geq tD, \]

where the expectation on the last line is over a uniformly chosen pair of distinct random codewords of \( C \). Hence,

\[ Z_0 \leq t(w - D/2). \]

We start with the lemma below.

*Lemma 2:* The sequence of random variables \( Z_i, i = 0, 1, \ldots, t \), forms a martingale.

The statement is true by construction. For completeness we present a proof that is deferred to Appendix A. Once we have proved that the sequence is a martingale, we show that it is a bounded-difference martingale.

*Lemma 3:* For any \( i = 1, \ldots, t \),

\[ |Z_i - Z_{i-1}| \leq (w - d/2) \left( 1 + \frac{t - i}{N - i} \right). \]

The proof is deferred to Appendix B.

Now using Azuma’s inequality for martingale with bounded difference [33], we have,

\[ \Pr \left( Z_t - Z_0 \geq \nu \right) \leq \exp \left( -\frac{\nu^2}{2(w - d/2)^2 \sum_{i=1}^{t+1} c_i^2} \right), \]

where, \( c_i = 1 + \frac{t - i}{N - i} \). This implies,

\[ \Pr \left( Z_t \geq \nu + t(w - D/2) \right) \leq \exp \left( -\frac{\nu^2}{2(w - d/2)^2 \sum_{i=1}^{t+1} c_i^2} \right). \]

Setting, \( \nu = w - t(w - D/2) \), we have,

\[ \Pr (Z^i \geq w) \leq \exp \left( -\frac{(w - t(w - D/2))^2}{2(w - d/2)^2 \sum_{i=1}^{t+1} c_i^2} \right). \]

Now,

\[ \sum_{i=1}^{t} c_i^2 \leq t \left( 1 + \frac{t - 1}{N - 1} \right)^2. \]

Hence,

\[ \Pr(\exists i \in [N] \setminus S : Z^i \geq w) \leq (N - t) \exp \left( -\frac{(w - t(w - D/2))^2}{2t(w - d/2)^2 \left( 1 + \frac{t - 1}{N - t} \right)^2} \right) < \epsilon, \]
when,
\[ \frac{d}{2} \geq w - \frac{w - t(w - D/2)}{\alpha \sqrt{t \ln \frac{N - t}{e}}} , \]
and \( \alpha \) is a constant greater than \( \sqrt{2} \left( 1 + \frac{t-1}{N-t} \right) \).

V. CONSTRUCTION

A. Discussions

As we have seen in Section II, constant-weight codes can be used to produce disjunct matrices. Kautz and Singleton [26] gives a construction of constant-weight codes that results in good disjunct matrices. In their construction, they start with a Reed-Solomon (RS) code, a \( q \)-ary error-correcting code of length \( q-1 \). For a detailed discussion of RS codes we refer the reader to the standard textbooks of coding theory [29], [38]. Next they replace the \( q \)-ary symbols in the codewords by unit weight binary vectors of length \( q \). The mapping from \( q \)-ary symbols to length-\( q \) unit weight binary vectors is bijective: i.e., it is \( 0 \rightarrow 100 \ldots 0; 1 \rightarrow 010 \ldots 0; \ldots ; q-1 \rightarrow 0 \ldots 1 \). We refer to this mapping as \( \phi \). As a result, one obtains a set of binary vectors of length \( q(q-1) \) and constant-weight \( q \). The size of the resulting binary code is same as the size of the RS code, and the distance of the binary code is twice that of the distance of the RS code.

For a \( q \)-ary RS code of size \( N \) and length \( q-1 \), the minimum distance is \( q-1 - \log_q N + 1 = q - \log_q N \). Hence, the Kautz-Singleton construction is a constant-weight code with length \( M = q(q-1) \), weight \( w = q-1 \), size \( N \) and distance \( 2(q - \log_q N) \). Therefore, from Prop. 3, we have a \( t \)-disjunct matrix with,
\[ t = \frac{q - 1 - 1}{q - 1 - q + \log_q N} = \frac{q - 2}{\log_q N - 1} \approx \frac{q \log q}{\log N} \approx \frac{\sqrt{M} \log M}{2 \log N} . \]

On the other hand, note that, the average distance of the RS code is \( (q-1)(1-1/q) \). Hence the average distance of the resulting constant-weight code from Kautz-Singleton construction will be
\[ D = \frac{2(q-1)^2}{q} . \]

Now, substituting these values in Theorem 2, we have a type-1 \( (t, \epsilon) \) disjunct matrix, where,
\[ \alpha \sqrt{t \ln \frac{N - t}{\epsilon}} \leq \frac{(q-t) \log q}{\log N} \approx \frac{(\sqrt{M} - t) \log M}{2 \log N} . \]
Suppose $t \leq \sqrt{M}/2$. Then,
\[ M(\ln M)^2 \geq 4\alpha^2 t(\ln N)^2 \ln \frac{2(N-t)}{e} \]
This basically restricts $t$ to be about $O(\sqrt{M})$. Hence, Theorem 2 does not obtain any meaningful improvement from the Kautz-Singleton construction in the asymptotics except in special cases.

**Example:** Consider a 4096-ary Reed-Solomon code of length 4095 and size $N = 4096^3 \approx 6.8 \times 10^{10}$ (number of elements). From the above discussion we see that, with number of tests $M = 4096 \cdot 4095 \approx 1.6 \times 10^7$, the resulting matrix is type-1 $(2700,2^{-4})$-disjunct. Although the number of defectives $t$ seems quite large here, it is very small compared to $N$. On the other hand the straight-forward Kautz-Singleton construction guarantees that for the same dimension of a matrix, we can have a $2047$-disjunct matrix. Roughly speaking, in this example it is possible to identify, by the merit of Theorem 2, $31.9\%$ more defective items, but the tests are successful in $93.75\%$ of the cases. It can be inferred that the improvement suggested in Theorem 2 appear only for very large values of $N$. However, in the asymptotic limits Kautz-Singleton construction is not optimal, as shown by the next construction.

There are two places where the Kautz-Singleton construction can be modified: 1) instead of Reed-Solomon code one can use any other $q$-ary code of different length, and 2) instead of the mapping $\phi$ any binary constant-weight code of size $q$ might have been used. For a general discussion we refer the reader to [13, §7.4]. In the recent work [37], the mapping $\phi$ is kept the same, while the RS code has been changed to a $q$-ary code that achieve the Gilbert-Varshamov bound [29], [38].

In our construction of disjunct matrices we follow the footsteps of [26], [37]. However, we exploit some property of the resulting scheme (namely, the average distance) and do a finer analysis that was absent from the previous works such as [37].

**B. $q$-ary code construction**

Next, we construct a *linear* $q$-ary code of size $N$, length $M_q$ and minimum distance $d_q$ that achieves the Gilbert-Varshamov (GV) bound [29], [38]. We describe the bound in Appendix C.

Porat and Rothschild [37] show that it is possible to construct in time $O(M_q N)$ a $q$-ary code that achieves the GV bound. To have such construction, they exploit the following well-known fact: a $q$-ary linear code with random generator matrix achieves the GV bound with high probability [38]. To have
an explicit construction of such codes, a derandomization method known as the method of conditional expectation [1] is used. In this method, the entries of the generator matrix of the code are chosen one-by-one so that the minimum distance of the resulting code does not go below the value prescribed by (6). For a detail description of the procedure, see [37].

Using the GV code construction of Porat and Rothschild and plugging it in the Kautz-Singleton construction above, we have the following proposition.

Proposition 4: Let $s \leq q$. There exists a polynomial time constructible family of $(M, N, 2M/q(1 - 1/s), M/q)$-constant-weight binary code that satisfy,

$$M/q \leq \frac{s \ln N}{\ln(q/s) - 1}. \quad (2)$$

Although the proof of the above proposition is essentially in Porat and Rothschild [37], we have a cleaner proof that we include in Appendix C for completeness.

However, we are also concerned with the average distance of the code. Indeed, we have the following proposition.

Proposition 5: The average distance of the code constructed by Prop. 4 is

$$D = \frac{2M}{q}(1 - 1/q).$$

Proof: For Prop. 4 we have followed the Kautz-Singleton construction. We take a linear $q$-ary code $C'$ of length $M_q \triangleq \frac{M}{q}$, size $N$ and minimum distance $d_q \triangleq \frac{d}{2}$. Each $q$-ary symbol in the codewords is then replaced with a binary indicator vector of length $q$ (i.e., the binary vector whose all entries are zero but one entry, which is 1) according to the map $\phi$. As a result we have a binary code $C$ of length $M$ and size $N$. The minimum distance of the code is $d$ and the codewords are of constant-weight $w = M_q = \frac{M}{q}$. The average distance of this code is twice the average distance of the $q$-ary code. As $C'$ is linear (assuming it has no all-zero coordinate), it has average distance equal to

$$\frac{1}{N} \sum_{j=1}^{M_q} jA_j = \frac{N}{N} \sum_{j=0}^{M_q} j \binom{M_q}{j} (1 - 1/q)^j (1/q)^{M_q - j} = M_q(1 - 1/q),$$

where $A_j$ is the number of codewords of weight $j$ in $C'$. Here we use the fact that the average of the distance between any two randomly chosen codewords of a nontrivial linear code is equal to that of a
binomial random variable [29]. Hence the constant-weight code $C$ has average distance $D = 2M_q(1 - 1/q)$.

C. Constructions for Model 1

We follow the Kautz-Singleton code construction. That is we have a $(M, N, d, M/q)$-constant-weight code that satisfy Prop. 4 and 5. Hence, average distance $D = 2M_q(1 - 1/q)$. The resulting test matrix will satisfy the condition of Thm. 1 when,

\[
d/2 \geq M/q - \frac{3(M/q - t(M/q - M/q(1 - 1/q)))^2}{2(2t(M/q - M/q(1 - 1/q)) + M/q) \ln N} \tag{3}
\]

or when,

\[
d/2 \geq M/q - \frac{3M/q(1 - t/q)^2}{2(2t/q + 1) \ln N}. \tag{4}
\]

Hence a sufficient condition is to chose the constant-weight code such that,

\[
d \geq \frac{2M}{q} \left(1 - \frac{3(1 - t/q)^2}{2(2t/q + 1) \ln N} \right).
\]

We can chose $q$ to be a smallest power of prime that is greater than $2t$. Which will make the sufficient condition look like,

\[
d \geq \frac{2M}{q} \left(1 - \frac{3}{16 \ln N} \right).
\]

However, according to Prop. 4, such code can be explicitly constructed with,

\[
M/q \leq \frac{16/3 \ln N \ln N}{\ln(3t/(16 \ln N/e)) - 1}. \tag{5}
\]

Hence, the total number of tests are $M = O\left(\frac{t}{\log t} \ln N \ln N \right)$.

D. Construction of almost disjunct matrix: Model 2

We choose $q$ to be a power of a prime number and write $q = \beta t$, for some constant $\beta > 2$. The value of $\beta$ will be chosen later. With the Kautz-Singleton construction with proper parameters we can have a disjunct matrix with the following property.
Theorem 3: Suppose $\epsilon > (N - t)e^{-\alpha t}$ for some constant $\alpha > 1$. It is possible to explicitly construct a $(t, \epsilon)$-disjunct matrix of size $M \times N$ where

$$M = O\left(t^{3/2} \ln N \frac{\sqrt{\ln \frac{N - t - \epsilon}{e}}}{\ln t}\right)$$

Proof: We follow the Kautz-Singleton code construction as earlier. That is we have a $(M, N, d, M/q)$-constant-weight code that satisfy Prop. 4 and 5. Hence, average distance $D = \frac{2M}{q}(1 - 1/q)$. The resulting matrix will be $(t, \epsilon)$-disjunct if the condition of Theorem 2 is satisfied, i.e.,

$$d/2 \geq M/q - \frac{M/q - t(M/q - M/q(1 - 1/q))}{\alpha \sqrt{t \ln \frac{N - t - \epsilon}{e}}} = M/q - \frac{M/q - \frac{tM/q}{q \alpha \sqrt{t \ln \frac{N - t - \epsilon}{e}}}}{\alpha \sqrt{t \ln \frac{N - t - \epsilon}{e}}} = \frac{M}{q} \cdot \left(1 - \frac{1 - \frac{t}{q}}{\alpha \sqrt{t \ln \frac{N - t - \epsilon}{e}}} \right).$$

Let us now use the fact that we have taken $q = \beta t$ to be a prime power for some constant $\beta$. Let us chose $\beta > e\alpha \sqrt{\alpha} + 1$.

Hence,

$$\frac{1 - \frac{t}{q}}{\alpha \sqrt{t \ln \frac{N - t - \epsilon}{e}}} = \frac{1}{\gamma \sqrt{t \ln \frac{N - t - \epsilon}{e}}},$$

for an absolute constant $\gamma = \frac{\alpha \beta}{\beta - 1}$. At this point, we can use Prop. 4, to have an explicit polynomial time construction of $(t, \epsilon)$-disjunct matrix with,

$$M \leq \beta t \ln N \frac{\gamma \sqrt{t \ln \frac{N - t}{e}}}{\ln \frac{\beta t}{\gamma \sqrt{t \ln \frac{N - t}{e}}} - 1}$$

$$= \beta \gamma t^{3/2} \ln N \frac{\sqrt{\ln \frac{N - t}{e}}}{\frac{1}{2}(\ln t - \ln \ln \frac{N - t}{e}) + \ln \frac{\beta}{\gamma} - 1}$$

$$= \beta \gamma t^{3/2} \ln N \frac{\sqrt{\ln \frac{N - t}{e}}}{\frac{1}{2}(\ln t - \ln \alpha) + \ln (e \sqrt{\alpha}) - 1}.$$
Note that the implicit constant in Theorem 3 is proportional to $\sqrt{a}$. We have not particularly tried to optimize the constant.

It is clear from Prop. 2 that a $(t, \epsilon)$ disjunct matrix is equivalent to a group testing scheme. Hence, as a consequence of Theorem 3, we will be able to construct a testing scheme with $O\left(t^{3/2} \log N \frac{\sqrt{\log \frac{N-t}{\epsilon}}}{\log t}\right)$ tests. Whenever the defect-model is such that all the possible defective sets of size $t$ are equally likely and there are no more than $t$ defective elements, the above group testing scheme will be successful with probability at least $1 - \epsilon$.

Note that, if $t$ is proportional to any positive power of $N$, then $\log N$ and $\log t$ are of same order. Hence it will be possible to have the above testing scheme with $O(t^{3/2} \sqrt{\log(N/\epsilon)})$ tests, for any $\epsilon > (N - t)e^{-t}$.

VI. CONCLUSION

In this work we show that it is possible to construct non-adaptive group testing schemes with small number of tests that identify a uniformly chosen random defective configuration with high probability. To construct a $t$-disjunct matrix one starts with the simple relation between the minimum distance $d$ of a constant $w$-weight code and $t$. This is an example of a scenario where a pairwise property (i.e., distance) of the elements of a set is translated into a property of $t$-tuples.

Our method of analysis provides a general way to prove that a property holds for almost all $t$-tuples of elements from a set based on the mean pairwise statistics of the set. Our method might be useful in many areas of applied combinatorics, such as digital fingerprinting or design of key-distribution schemes, where such a translation is evident. With this method new results may be obtained for the cases of cover-free codes [15], [26], [41], traceability and frameproof codes [10], [40].

APPENDIX

A. Proof of Lemma 2

We have created a sequence here that is a martingale by construction. This is a standard method due to Doob [11], [33]. Let,

$$w - \frac{d_H(c_i, c_j)}{2} = Y_i.$$
Consider the \( \sigma \)-algebras \( \mathcal{F}_k \), \( k = 0, \ldots, t + 1 \), where \( \mathcal{F}_0 = \{\emptyset, [N]\} \) and \( \mathcal{F}_k \) is generated by the partition of the set of \( \binom{N}{t+1} \) possible choices for \((t + 1)\)-sets into \( \binom{N}{k} \) subsets with the fixed value of the first \( k \) indices, \( 1 \leq k \leq t + 1 \). The sequence of increasingly refined partitions \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{t+1} \) forms a filtration such that \( Z_k \) is measurable with respect to \( \mathcal{F}_k \) (is constant on the atoms of the partition).

We have,

\[
Z_i = \mathbb{E} \left( \sum_{j=1}^{t} Y_j \mid Y_1, \ldots, Y_i \right)
\]

\[
= \sum_{j=1}^{i} Y_j + \mathbb{E} \left( \sum_{j=i+1}^{t} Y_j \mid Y_1, \ldots, Y_i \right)
\]

\[
= Z_{i-1} + Y_i + \mathbb{E} \left( \sum_{j=i+1}^{t} Y_j \mid Y_1, \ldots, Y_i \right) - \mathbb{E} \left( \sum_{j=1}^{t} Y_j \mid Y_1, \ldots, Y_{i-1} \right).
\]

We therefore have,

\[
\mathbb{E} \left( Z_i \mid Z_0, \ldots, Z_{i-1} \right) = Z_{i-1} + \mathbb{E} \left( Y_i \mid Z_0, \ldots, Z_{i-1} \right)
\]

\[
+ \mathbb{E} \left( \sum_{j=i+1}^{t} Y_j \mid Y_1, \ldots, Y_i \right) \mid Z_0, \ldots, Z_{i-1} \right)
\]

\[
- \mathbb{E} \left( \sum_{j=1}^{t} Y_j \mid Y_1, \ldots, Y_{i-1} \right) \mid Z_0, \ldots, Z_{i-1} \right)
\]

\[
= Z_{i-1} + \mathbb{E} \left( Y_i \mid Z_0, \ldots, Z_{i-1} \right)
\]

\[
+ \mathbb{E} \left( \sum_{j=i+1}^{t} Y_j \mid Z_0, \ldots, Z_{i-1} \right) - \mathbb{E} \left( \sum_{j=i}^{t} Y_j \mid Z_0, \ldots, Z_{i-1} \right)
\]

\[
= Z_{i-1}.
\]

**B. Proof of Lemma 3**

Let us again assume that,

\[
w - \frac{d_H(c_1, c_j)}{2} = Y_j.
\]

We have,

\[
|Z_i - Z_{i-1}|
\]
\[
\sum_{j=1}^{t} Y_j \mid Y_1, \ldots, Y_{i-1}, Y_i \} - \mathbb{E} \left( \sum_{j=1}^{t} Y_j \mid Y_1, \ldots, Y_{i-1}, Y_i = b \right) \right| \\
\leq \max_{0 \leq a, b \leq w-d/2} \left| \sum_{j=1}^{t} \left( \mathbb{E} (Y_j \mid Y_1, \ldots, Y_i = a) - \mathbb{E} (Y_j \mid Y_1, \ldots, Y_{i-1}, Y_i = b) \right) \right| \\
= \max_{0 \leq a, b \leq w-d/2} \left| a - b + \sum_{j=i+1}^{t} \left( \mathbb{E} (Y_j \mid Y_1, \ldots, Y_i = a) - \mathbb{E} (Y_j \mid Y_1, \ldots, Y_i = b) \right) \right| \\
\leq \max_{0 \leq a, b \leq w-d/2} \left| w - d/2 + \sum_{j=i+1}^{t} \left[ \mathbb{E} \left( \frac{w - d_H(c_1, c_j)}{2} \mid d_H(c_1, c_1), \ldots, d_H(c_1, c_i) = 2(w - a) \right) - \mathbb{E} \left( \frac{w - d_H(c_1, c_j)}{2} \mid d_H(c_1, c_1), \ldots, d_H(c_1, c_i) = 2(w - b) \right) \right] \right| \\
= (w - d/2) \left( 1 + \frac{t-i}{N-1} \right) \\
= (w - d/2)c_i,
\]

where \( c_i = 1 + \frac{t-i}{N-1} \).

C. Gilbert-Varshamov bound and proof of Prop. 4

Lemma 4 (Gilbert-Varshamov Bound): There exists an \((m, N, d)_{q}\)-code such that,

\[
N \geq \frac{2^m}{\sum_{i=0}^{d-1} \binom{m}{i} (q-1)^i}.
\]  

(6)

Corollary 1: Suppose \( X \) is a Binomial\((m, 1 - \frac{1}{q})\) random variable. There exists an \((m, N, d)_{q}\)-code such that,

\[
N \geq \frac{1}{\Pr(X \leq d)}.
\]

Lemma 5: Suppose \( X \) is a Binomial\((m, 1 - \frac{1}{q})\) random variable. Then, for all \( s < q \),

\[
\Pr \left( X \leq m \left( 1 - \frac{1}{s} \right) \right) \leq e^{-mD \left( 1/s \right) ||1/q)},
\]

(7)
where $D(p||p') = p \ln(p/p') + (1-p) \ln((1-p)/(1-p'))$.

Theorem 4: Let $s < q$. For the $(m, N, m(1 - 1/s))_q$-code that achieves the Gilbert-Varshamov bound, we have

$$m \leq \frac{s \ln N}{\ln(q/s) - 1}. \quad (8)$$

Proof: This theorem follows from corollary 1 and lemma 5. Note that,

$$D(1/s||1/q) = \frac{1}{s} \ln \frac{q}{s} + \left(1 - \frac{1}{s}\right) \ln \left(1 - \frac{1}{s}\right) - \left(1 - \frac{1}{q}\right) \ln \left(1 - \frac{1}{q}\right) \geq \frac{1}{s} \ln \frac{q}{s} - \frac{1}{s} ,$$

where in the last line we have used the fact that $x \ln x \geq x - 1$ for all $x > 0$.

Using the Kautz-Singleton construction, this implies that, there exists a polynomial time constructible family of $(M, N, 2M/q(1 - 1/s), M/q)$-constant-weight binary code with,

$$M/q \leq \frac{s \ln N}{\ln(q/s) - 1},$$

which is Prop. 4.

Acknowledgements: The author would like to thank Alexander Barg for many discussions related to the group testing problem.

REFERENCES


